Advances in Applied Mathematical Biosciences.
ISSN 2248-9983 Volume 2, Number 1 (2011), pp. 27-38

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An Optimal Control Problem on Temperature Distribution in Tissue by Induced Microwave

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Abstract

In this paper, an optimal control problem described by one-dimensional bio-heat transfer equation in a single layered (homogeneous) tissue is analytically investigated such that a beneficial therapeutic (desired) temperature on some specific points along the entire length of the tumour inside the tissue can be attained during the specific time by controlling optimally heating power induced by microwave applied on the surface of the tissue [13] when the surface cooling temperature is taken as constant throughout the fixed operation of the process. Here, the methodology adopted is conjugate gradient method under calculus of variation.

Numerical calculation of temperature distribution along the length of the tissue on various values of operation of the process are carried out.

Keywords: Optimal control, microwave, cooling temperature, tumour, hyperthermia.

Nomenclature

c = specific heat of tissue, J/kg ^{o}C
h = heat transfer coefficient between the skin and the ambient air, $Wm^{-2}/^{0}C$
k = thermal conductivity of tissue, $W\,m^{-1}/^{0}C$
L = length of the tissue m

χ_i = Locations of tumor, m
χ = temperature, ^{0}C
χ_a = arterial temperature, ^{0}C
χ_0 = initial temperature, ^{0}C
u(t) = temperature of the surrounding medium, ^{0}C
χ^* = desired temperature to be attained, ^{0}C
T = Total time of the process, s
t_1 = switching time, s
Q(t) = Heating power induced by microwave(Wm^{-3},).
ρ = density of tissue, kg m^{-3}
δ = dirac-delta function.
ω = product of flow and heat capacity of blood, W m^{-3} / ^{0}C
Q_m = rate of metabolic heat generation, Wm^{-3}

Introduction

After the modern applications of Maximal Principle, one probably can recognize that optimal control theory can be treated within the framework of calculus of variation. Because the optimal control theory had reached a good degree of stability and perfection, it is believed that a thorough and careful presentation of the current status of optimal control theory will serve the useful purpose of offering primarily a foundation where later researches can be based, particularly in biological systems.

An important class of problem with distributed parameters are the problems of optimal heating of tissue. Modeling and understanding heat transport and temperature variation within biological tissue are key issues in medical thermal therapeutic applications, such as hyperthermia cancer treatment.

In hyperthermia treatments the goal is to rise the temperature at the location of the tissue affected by tumour to it’s therapeutic value while avoiding the damage of the healthy tissue which is generally performed with the aid of simulations for the purpose of determining the optimum power of heat sources and surface cooling temperature as these are accessible direct control input variables [16].

Deng et al [1] investigated analytically the problems in bio-heat transfer equation by heat source applied on the skin surface or inserting a heating probe at the tumor site with the aid of Green’s function. An analytical investigation was carried out by Dhar et al [2] on optimal temperature control in hyperthermia where the control of surface cooling was considered.

Butkovasky [3] studied the fundamentals of optimal control problems in distributed system. With the aid of finite element method Das et al [4] developed an analytical investigation on the aspect of computation technique for fast temperature optimization in hyperthermia.

Kowalski et al [5] conducted a study on cost minimization problem in space by feedback control system applying electro-magnetic phased-array.

Loulou et al [6] analyzed analytically a thermal dose optimization problem in hyperthermia using conjugate gradient method under calculus of variation. The study of Salloum et al [7] developed an optimization algorithm of heat absorption pattern in the treatment planning by applying magnetic nanoparticle injections in tumour.

In course of investigation on optimization of radio-immunotherapy (RIT) interactions with hyperthermia, the combination of local hyperthermia with RIT has been discussed in Kinuya et al [8]. Szasz et al [9] proposed a generalization of the entire energy balance where the new paradigm could be a theoretical basis of the empirical dose-construction for oncological hyperthermia.

The presentation of a PDE-constrained optimization algorithm in Schenek et al [10] carries considerable importance in hyperthermia cancer treatment planning. The article of Rapoport et al [11] described the study of targeted chemotherapeutic intervention on solid tumors by means of ultrasound. Wager in [12] investigated a computer simulation to calculate transient temperature distributions in realistic cross sections of the human body.

Kuznetsov in [13] investigated optimal control problem to maximize the temperature in the tumor at the end of time of operation of the process due to spatial volumetric heat generation by assuming fixed total volumetric heat generation over the duration of the process. With the aid of conjugate gradient method, a distributed optimal problem of a system described by bio-heat equation in a homogeneous plane tissue due to induced microwave was investigated by Dhar et al [14]. Cheng et al [15] studied heating systems with large number of physical sources for temperature optimization in hyperthermia by finite difference method. The major study in [15] was to approximate the heating pattern of a large number of physical sources(antennas) with a smaller number of pre-calculated linearly independent source configurations, i.e., virtual sources which would provide a sufficiently goal optimization solution. The temperature distribution in the three-dimensional tissue was investigated by steady state bio-heat equation .

Wagtar[16] studied simulation-oriented optimization method to determine the input control variables so as to achieve specified temperature of the tumour inside the tissue in an optimum manner. An analytical investigation of heat transport in biological tissue in hyperthermia was carried out by Mahjoob et.al [17].

A typical treatment with local hyperthermia consists of raising the temperature of the tumour to about $40-43^0c$ by avoiding rise of the temperature of the healthy tissue to 40^0c [14,15,16, 17]. Therefore, an optimal treatment goal is to be assumed so that all tumour temperatures, located at some specific points along the entire length of the tumour embedded inside the tissue , should attain a specific therapeutic beneficial (desired) temperature 43^0c [15]. This assumption attempts to uniformize the temperature of all tumours, located at specific points along it's entire length to a therapeutic desired temperature 43^0c [15].

In this paper, an optimal control problem in distributed parameter system described by one-dimensional bio-heat equation in a single layered (homogeneous) tissue is analytically investigated such that a beneficial therapeutic (desired) temperature on some specific points along the entire length of the tumour embedded inside the tissue can be attained during specific time by controlling optimally time-dependent microwave induced heating power $Q(t)\,(Wm^{-3})$ where surface cooling temperature is taken as constant throughout the fixed operation of the process.

In course of analytical investigation of the problem, it is found that the optimal control variable $Q(t)$ (Wm^{-3}) is a singular control and thus $Q(t)$ (Wm^{-3}) changes it's value at certain specific discrete instances designated as switching time. For the sake of simplicity, one switching time t_1 (say) is considered and so $Q(t)$ (Wm^{-3}) assumes two extreme values within the intervals $(0, t_1)$ and (t_1, T) where T is the total time of operation of the process [3,6,14].

The objective of this paper is to obtain the optimal value of the control $Q(t)$ (Wm^{-3}) for specified value of t_1 by conjugate gradient method under calculus of variation [3,14].

With the obtained values of $Q(t)$ (Wm^{-3}), numerical temperature distributions of the tissue at different times on various values of total time of operation of the process are carried out which display the desired temperature of the tumours.

Mathematical Analysis

The one dimensional Pennes bio-heat equation [1,2] can be written as,

$$\rho c \frac{\partial \chi}{\partial t} = k \frac{\partial^2 \chi}{\partial x^2} + \omega(\chi_a - \chi) \ + + Q(t) + Q_m \tag{1}$$

Boundary condition

$$k \frac{\partial \chi}{\partial x} = h\{\chi - u(t)\} \ on \ x = 0$$

$$\chi = \chi_a \ on \ x = L \tag{2}$$

Initial condition

$$\chi(x, o) = \chi_O \tag{3}$$

We would like to attain the therapeutic beneficial (desired) temperature χ^* on q number of tumours , located at specific points $x = x_1, x_2, \ldots\ldots\ldots x_q$ along the entire length of the tumour inside the tissue, during the specific time by controlling optimally microwave induced heating power applied on the surface of the tissue when the total time of operation of the process T is fixed. The objective function (functional) can be written as, [15,16]

$$\frac{1}{2}\left[\frac{1}{q}\int_0^T\int_0^L \sum_{i=1}^{q} \{\chi^* - \chi(x,t)\}^2 \delta\{x - x_i\} dx dt\right] \tag{4}$$

Which is taken as performance criterion to be minimized.

Here, $\chi(x,t)$, T, L and δ designate the temperature of the tissue , total time of opration of the process, length of the tissue and dirac-delta function respectively.

The performance criterion, given by equation (4), signifies the ratio of the sum of the square deviations of the desired temperature χ^* from the temperature on q-number of tumours, located at specific points $x = x_1, x_2,.......x_q$ along it's entire length inside the tissue, to the q number of tumours.

Let us write, a function J, given by, [6,14]

$$J = -\frac{1}{2}\left[\frac{1}{q}\int_0^T\int_0^L\sum_{i=1}^{q}\left\{\chi^* - \chi(x,t)\right\}^2 \delta\{x - x_i\}dxdt\right]$$

$$+\int_0^L\int_o^T \psi(x,t)\left\{\frac{k}{\rho c}\frac{\partial^2\chi}{\partial x^2}+\frac{\omega}{\rho c}(\chi_a - \chi)+\frac{1}{\rho c}Q(t)+\frac{Q_m}{\rho c}-\frac{\partial}{\partial t}\chi\right\}dxdt \tag{5}$$

Where $\psi(x,t)$ is the adjoint function .

The first variation of the functional J can be written as δJ for small change $\delta\chi(x,t)$ of $\chi(x,t)$, where $\chi(x,t)$ receives a small change $\delta\chi(x,t)$ due to the change of the control variable $Q(t)$.

In order to obtain the optimality condition, i.e., the stationary condition $\delta J = 0$, we have developed the expressions , given by,

$$\int_0^L\int_0^T \psi(x,t)\frac{\partial^2}{\partial x^2}\delta\chi(x,t)\ dx\ dt =$$

$$\int_0^T\left[\frac{h}{k}\{\psi(0,t)\delta u(t) - \psi(0,t)\delta\chi(0,t)\} - \frac{\partial\psi(L,t)}{\partial x}\delta\chi(L,t) + \psi(L,t)\delta\chi(L,t) + \frac{\partial\psi(0,t)}{\partial x}\delta\chi(0,t)\right]dt$$

$$+\int_0^L\int_0^T \frac{\partial^2}{\partial x^2}\psi(x,t)\delta\chi(x,t)\ dx\ dt \tag{5a}$$

and

$$\int_0^L\int_0^T \psi(x,t)\frac{\partial}{\partial t}\delta\chi(x,t)\ dx\ dt = \int_0^L\{\psi(x,T)\delta\chi(x,T) - \psi(x,0)\delta\chi(x,0)\}dx -$$

$$\int_0^L\int_0^T \frac{\partial\psi(x,t)}{\partial t}\delta\chi(x,t)\ dx\ dt \tag{5b}$$

by using integration by parts with the help of equations (2) and (3).

$$\delta J = \left[\frac{1}{q}\int_0^T\int_0^L\sum_{i=1}^{q}\left\{\chi^* - \chi(x,t)\right\}\,\delta\{x - x_i\}dxdt\right]$$
$$+\frac{k}{\rho c}\int_o^T \psi(L,t)\delta\chi_x(L,t)dt + \frac{1}{\rho c}\int_o^T\left\{k\frac{\partial\psi}{\partial x}(o,t) - h\psi(o,t)\right\}\delta\chi(o,t)dt$$
$$+\frac{h}{\rho c}\int_o^T \psi(o,t)\delta u(t)dt - \frac{k}{\rho c}\int_o^T\frac{\partial}{\partial x}\psi(L,t)\delta\chi(L,t)dt + \frac{k}{\rho c}\int_o^T\int_o^L\frac{\partial^2}{\partial x^2}\psi(x,t)\delta\chi(x,t)dxdt$$
$$-\frac{\omega}{\rho c}\int_o^L\int_o^T\psi(x,t)\delta\chi(x,t)dxdt + \frac{1}{\rho c}\int_o^T\int_o^L\psi(x,t)\delta(x - x_1)\delta Q(t)dxdt$$
$$+\int_0^L\int_o^T\frac{\partial\psi(x,t)}{\partial t}\delta\chi(x,t)dxdt - \int_o^L\psi(x,T)\delta x(x,T)dx$$
$$+\int_0^L\psi(x,o)\delta\chi(x,o)dx \tag{6}$$

with the help of equations (2) and (3). By assuming δJ to vanish for any $\delta\chi_x(L,t), \delta\chi(x,t), \delta\chi(o,t), \delta\chi(x,T), \delta Q(t)$, $\delta u(t)$ and taking $\delta\chi(x,o), \delta\chi(L,t)$ both equal to Zero, a system of adjoint function $\psi(x,t)$ is obtained as,

$$\frac{\partial\psi}{\partial t} + \frac{k}{\rho c}\frac{\partial^2\psi}{\partial x^2} = \left[-\frac{1}{q}\sum_{i=1}^{q}\left\{\chi^* - \chi(x,t)\right\}\,\delta\left\{x - x_i\right\}\right] + \frac{\omega}{\rho c}\psi. \tag{7}$$

$$k\frac{\partial\psi}{\partial x} = h\psi \;\; on\, x = 0 \tag{8}$$

$$\psi(x,t) = 0\, on\, x = L$$

$$\psi(x,T) = 0 \tag{9}$$

and the optimal value of the controls Q(t) and u(t) stand,

$$Q(t) = Sign\frac{1}{\rho c}\int_0^L\psi(x,t)dx \tag{10}$$

$$u(t) = Sign\,\psi(o,t), \tag{11}$$

Here the conjugate gradient method with the aid of calculus of variation have been used [3,6,14]. Considering $\chi_1(x,t) = \chi\;(x,t) - \chi_a$ and expressing $\chi_1(x,t)$ in Finite Sine Transform, given by,

$$\overline{\chi}_{1n}(t) = \int_0^L\chi_1(x,t)\sin p_n(L - x)dx \tag{12}$$

and

$$\chi_1(x,t) = \sum_{n=1}^{\infty} \overline{\chi}_{1n}(t) \times \frac{2\sin p_n(L-x)}{L - \dfrac{\sin 2p_n L}{2p_n}} \tag{13}$$

where p_n are positive, real roots of the equation,

$$p\,cot(pL) = \frac{-h}{k} \tag{14}$$

the equation (1) with the help of equations (1.2), (1.3) and (1.13) stands,

$$\frac{d}{dt}\overline{\chi}_{1n}(t) + \alpha_{1n}\overline{\chi}_{1n}(t) = \alpha_{3n}Q(t) + \alpha_{4n} + \alpha_{5n};\ n = 1,2,3,\cdots \tag{15}$$

Where,

$$\begin{aligned}
\alpha_{1n} &= \frac{1}{\rho c}\left\{kp_n^2 + \omega\right\},\\
\alpha_{4n} &= \frac{h}{\rho c}\left\{u(t) - \chi_a\right\}\sin p_n L,\\
\alpha_{3n} &= \frac{1}{\rho c}\left(\frac{1-\cos p_n L}{p_n}\right)Q_m\\
\alpha_{5n} &= \frac{1}{\rho c}\left(\frac{1-\cos p_n L}{p_n}\right)Q_m
\end{aligned} \tag{16}$$

Finally we get,

$$\chi(x,t) = \chi_a + \sum_{n=1}^{\infty} \overline{\chi}_{1n}(t) \times R_n(x) \tag{17}$$

The solution of equation (15) with the help of equation (16) stands,

$$\begin{aligned}
\overline{\chi}_{1n}(t) = {}&[(\chi_o - \chi_a)\left(\frac{1-\cos p_n L}{p_n}\right) + \frac{h}{\rho c}\sin p_n L \int_0^t \{u(\xi) - \chi_a\} e^{\alpha_{1n}\xi} d\xi + \left(\frac{1-Cos p_n L}{p_n}\right)\frac{1}{\rho c} Q_m \int_0^t e^{\alpha_{1n}\xi} d\xi \\
&+ \left(\frac{1-Cos p_n L}{p_n}\right)\frac{1}{\rho c}\int_0^t Q(\xi) e^{\alpha_{1n}\xi} d\xi] \times e^{-\alpha_{1n}t};\ n = 1,2,3,\cdots
\end{aligned} \tag{18}$$

where $$R_n(x) = \frac{2\sin p_n(L-x)}{L - \dfrac{\sin 2p_n L}{2p_n}} \tag{19}$$

The corresponding solution of equation (7) with the help of equations (8) and (9) can be written as, with the help of earlier Finite Transform,

$$\psi(x,t) = \sum_{m=1}^{\infty} \overline{\psi}_m(t)\, R_m(x) \tag{20}$$

where

$$\overline{\psi_m}(t) = \frac{1}{q}\left[\int_T^t \sum_{i=1}^{q} \left\{ \left(\chi_a - \chi^*\right) + \sum_{n=1}^{\infty} \overline{\chi_{1n}}(\tau) \times R_n(x_i) \right\} \times \sin p_m (L - x_i) \times e^{-\alpha_{1m}\tau} d\tau \right] \times e^{\alpha_{1m} t} \tag{21}$$

for p_m are roots of the equation (14).

Here, from equations (10)-(11), we note that $Q(t)$ and $u(t)$ are singular controls.

Considering u(t) as constant, the value of optimal control Q(t) can be obtained from equation (10) with the help of equations (14-21).

It is assumed that the time dependent controllable input Q(t) (Wm^{-3}) is piecewise constant function of time that changes value at certain specified discrete instants considered as switching times [12,14,16].

For the sake of simplicity we consider only one specified switching time $t = t_1$

Thus according to equation (10) one can write

$$Q(t_1) = \int_0^L \psi(x, t_1) = 0 \tag{22}$$

where Q(t) assumes two extreme values in $(0, t_1)$ and (T, t_1) which can be obtained with the help of equations (14-21) by means of simulation.

Numerical Results

Data used in computation are given as follows [13]

C	=	3770 J kg^{-1} ^{0}C^{-1}
ρ	=	998 kgm^{-3}
K	=	.5 Wm^{-1} ^{0} C^{-1}
H	=	6 Wm^{-2} ^{0}C^{-1}
χ_a	=	37^{0}C
χ^*	=	43^{0}C
L	=	.01 m,
x_1	=	.006m
ω	=	3000 Wm^{-3} ^{0} C^{-1}
Q_m	=	33800 Wm^{-3}
χ_0	=	25^{0}C
T	=	600s, 800s, 1000s
u(t)	=	20^{o}C

All the results are obtained by taking five specific points on the tumour of length .001m , lies in .005m $\leq x_i \leq$ .006, with the help of computer simulation.

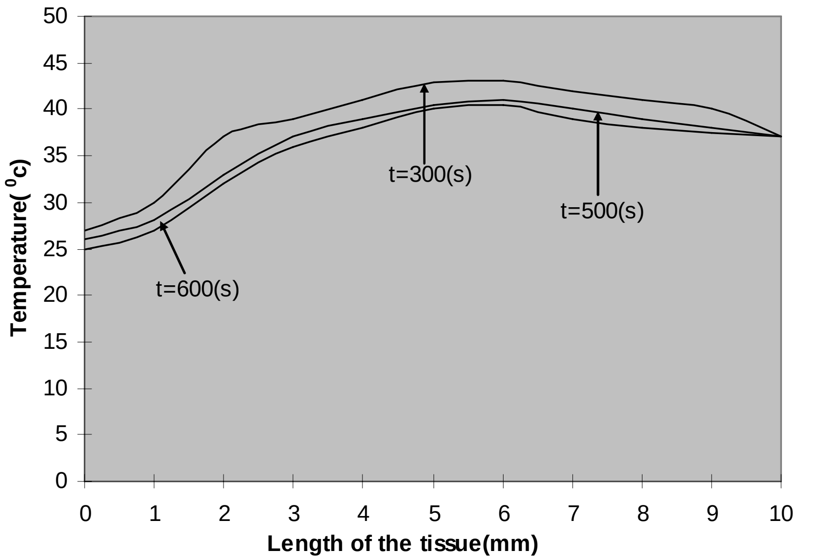

Fig 1 : Temperature(^{0}c) of the tissue along it's length (mm) for Q(t)=408493 wm^{-3}, 0<=t<=300; Q(t)=0, 300< =t<=600;

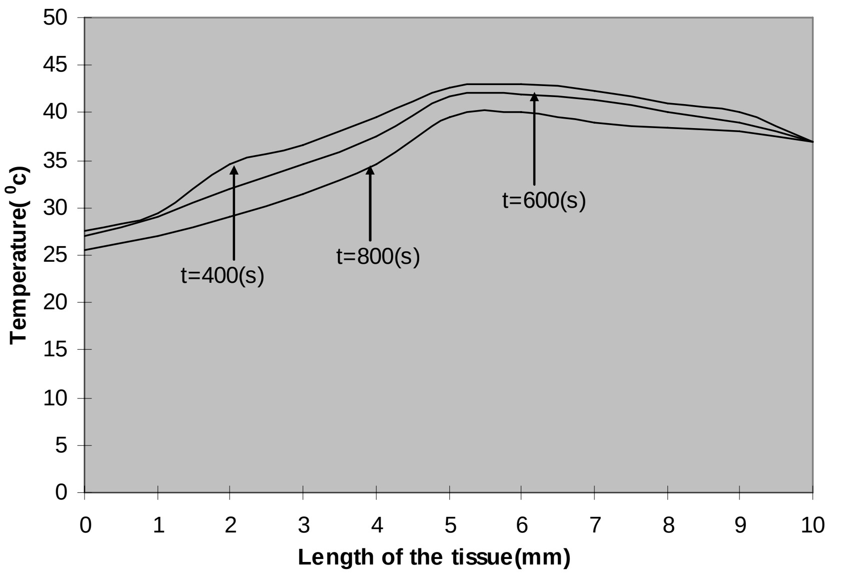

Fig 2 : Temperature(^{0}c) of the tissue along it's length (mm) for Q(t)=380121 wm^{-3}, 0<=t<=400; Q(t)=0, 400< =t<=800;

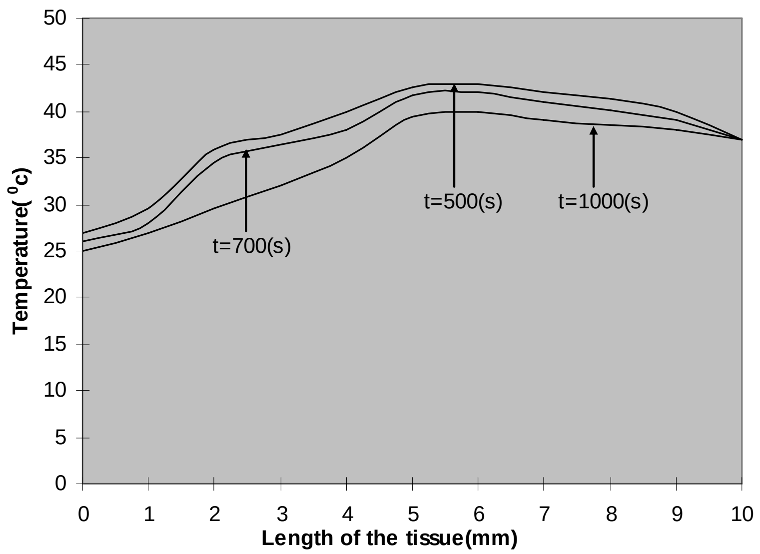

Fig 3 : Temperature(^{0}c) of the tissue along it's length (mm) for Q(t)=358191 wm^{-3}, 0<=t<=500; Q(t)=0, 500< =t<=1000;

Fig 1 depicts the temperature distribution along the length of the tissue subject to Q(t) = 408493 Wm^{-3}, $0 \le t \le 300$; Q(t) = 0, $300 \le t \le 600$ at different times for total time of operation T = 600S. It is observed that the tumour temperature along it's entire length within $.005 - .006$ m attains desired temperature $43^{0}c$ at t = 300s (Switching time). In Fig 2 it is shown that the temperature of the tumour along it's length within $.005 - .006$ m reaches desired temperature $43^{0}c$ at switching time to t = 400s due the application of volumetric heat generation rate Q(t) = 380121 Wm^{-3}, $0 \le t \le 400$; Q(t) = 0, $400 \le t \le 800$ for the total time of operation of the process T = 800s. Fig 3 displays the temperature of the tissue along it's length at different times for total time of operation of the process T = 1000s where it is found that the temperature of the tumour along it's entire length within $.005 - .006$ m rises $43^{0}c$ at the switching time t = 500s when heat source Q (t) = 358191 Wm^{-3}, $0 \le t \le 500$; Q(t) = 0, $500 \le t \le 1000$ is being applied.

It has been observed that distributions of temperature of the tissue on the left side of the tumour, which lies between .005m to .006 m, are always less than $43^{0}c$ of the tumour. Further the temperature on the right side of the tumour decreases steadily to $37^{0}c$ (arterial temperature) which can be accounted for as the effect of cutting off the volumetric heat generation rate Q(t) (Wm^{-3}) in the second time segment of operation of the process. Thus the damage of normal tissue due to overheating is avoided. It is further noteworthy to mention that as the total time of operation of the process increases from T = 600s to 1000s, the time in the first segment of operation increases with corresponding decrease of Q(t) (Wm^{-3}) in this segment.

Conclusion

Numerical results on temperature distribution of the tissue, which displays the rise of the temperature of the tumour at different times of operation of the process, may be useful for the purpose of realistic computer aided therapy planning in hyperthermia treatment.

It can further be developed at different points of location of the tumour alongwith different length of the tissue concerned which may focus a useful guideline to illustrate the versatility of the computer program.

Again as surface cooling temperature is one of the direct control input variable, this analytical study can be applied taking into consideration the optimum surface cooling temperature alongwith optimum spatial heating power induced by microwave.

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