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A Predator–Prey Model with Discrete Time Delay Considering Different Growth Function of Prey

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Abstract

The dynamics of a predator -prey interaction model is studied here considering different growth functions of prey and including a discrete time delay to model the time lags between the capture of the prey and its conversion to viable biomass. Linear stability analysis reveals that in the absence of delay for the monotonic growth function of prey, the coexistence equilibrium is a centre, but if the growth function of prey is logistic, then the coexistence equilibrium is locally asymptotically stable if $d < \beta K$ and it does not exist if $d > \beta K$. It is shown that if $\tau = \tau_0 > 0$, periodic solution arises in case of the monotonic growth function of prey as Hopf bifurcation occurs without any condition. In the case of the logistic growth of prey when $\tau = \tau_0 > 0$, the periodic solution is possible through Hopf bifurcation under certain conditions.

Keywords: Delays, Biomass, Stability, Periodic Solution, Hopf bifurcation.

Introduction

In the real world, the biosphere is an important zone for biological activities that are mainly responsible for the changes in ecology and environment and the growth rate of different species mainly depend on ecology, carrying capacity of environment etc. As a consequence growth rate of the prey pieces is an important matter for the predator-prey interaction model. The co-existence of interacting biological species has been of great interest in the past few decades and has been studied extensively using mathematical models by several researchers [1, 2, 3, 4, 5, 6]. A predator-prey model without delay was studied by Dubby [6] considering different growth of predator. Following [6], different growth functions of prey population are considered here. In many existing predator-prey ODE models [6, 13, 14, 15], the time delay for the conversion of biomass from prey to predator population were ignored. The reality is that in the predator equation, the delay is often caused by the conversion of consumed prey biomass in to the predator biomass, may it be in the form of body size growth or reproduction. Fan and Wolkowicz [8] studied a predator-prey model in the chemostat with discrete time delay. Following [6, 7, 8, 12], a predator prey model is proposed including a discrete time delay to model the time lags between the capture of the prey and its conversion to viable biomass , considering different growth functions of prey. Moreover, the term $e^{-\delta\tau}$ is included in predator equation which accounts for predators those interact with prey at time t but die before giving reproduction (or growth) τ time units later (i.e. if we assume a constant death rate δ for those predators that survive in gestation period that means the probability of surviving between the time lags for converting biomass).

In population dynamics, a functional response of the predator to the prey density refers to the change in the density of prey attached per unit time per predator as the prey density changes. For simplicity, Holling type I form (i.e. $h(x(t)) = \beta\, x(t)$, $\beta > 0$) of functional response is considered for both cases. The main purpose of this, study is to analyze the dynamics of the predator –prey interacting population model due to different growth function, including discrete time delay for the capture of the prey and its conversion to biomass and the term $e^{-\delta\tau}$. Analyses are shown for monotonic growth function of prey in section -1 and logistic growth function of prey in section -2.

Model formulation

The proposed model is

$$\frac{dx(t)}{dt} = g(x(t)) - h(x(t))\; y(t)$$

$$\frac{dy(t)}{dt} = e^{-\delta\tau}\; h(x(t-\tau))\; y(t-\tau) - d\, y(t) \tag{1}$$

Subject to the following initial conditions:

$$x(\theta) = \phi_1(\theta) \geq 0, \quad \theta \in [-\tau, 0), \; \phi_1(0) > 0$$

$$y(\theta) = \phi_2(\theta) \geq 0, \quad \theta \in [-\tau, 0), \; \phi_2(0) > 0 \tag{2}$$

Here $x(t)$ denote the density of prey population, $y(t)$ is the density of the predator population, $g(x(t))$ denote the growth function of prey population and $h(x(t))$ denote the functional response of the predator on prey, and d is the death rate of the predator population. Assume that the growth rate of predator depends only on the prey population. Two growth functions for the prey population are,

(i) $g(x(t)) = r\, x(t)$, $r > 0$

(ii) $g(x(t)) = r\,x(t)\,(1 - \frac{x(t)}{K}), \quad r > 0, \; K > 0,$

where K is the carrying capacity of the environment.

Section-1

Considering the monotonic growth function of prey and Holling type I, functional response for predator, the Model (1) becomes

$$\frac{dx(t)}{dt} = r\,x(t) - \beta\,x(t)\,y(t)$$
$$\frac{dy(t)}{dt} = \beta\,e^{-\delta\tau}\,x(t-\tau)\,y(t-\tau) - d\,y(t) \tag{1.1}$$

Positivity of the solution

It is important to show positivity for the system (1.1) as they represent prey- predator populations. Biologically, positivity implies that the population survives. For proof this, following Zhu and Zou [9], we have the following theorem:

Theorem1.1. Let $(\phi_1(\theta), \phi_2(\theta)) \in C([-\tau, 0], \Re_+^2)$ and $(x(t), y(t))$ be any solution to system (1.1) with the initial conditions (2). Then we have the following:

$$x(t) > 0, y(t) > 0 \text{ for } t > 0.$$

Proof: To prove $x(t) > 0$ for $t \in [0, \infty)$, from the first equation in (1.1), it follows that

$$\frac{dx(t)}{dt} = (r - \beta y(t))\,x(t)$$

$$\Rightarrow \frac{dx(t)}{x(t)} = (r - \beta y(t))\,dt$$

$$\Rightarrow x(t) = x(0)\exp\left(\int_0^t (r - \beta\,y(t))dt\right) > 0, \quad \forall\, t > 0$$

$$\Rightarrow x(t) > 0$$

Now to prove that $y(t) > 0$ on $[0, \infty)$, suppose that there exists $\bar{t} > 0$ such that $y(\bar{t}) = 0$, and $y(t) > 0$ for $t \in [0, \bar{t})$. Then $\dot{y}(\bar{t}) \le 0$, [8]. From the second equation of (1.1), we have

$$\dot{y}(\bar{t}) = \beta e^{-\delta\tau} x(\bar{t} - \tau)\,y(\bar{t} - \tau) - d\,y(\bar{t})$$

$$\dot{y}(\bar{t}) = \beta e^{-\delta\tau} x(\bar{t} - \tau)\,y(\bar{t} - \tau) > 0 \;,$$

a contradiction, therefore $y(t) > 0\,, \forall\, t \in [0\,, \infty)$.

Equilibria and Stability Analysis

The Model (1.1) has two equilibrium points: $E_1 = (0\,,0)$ is a trivial equilibrium which is biologically meaningless and $E_2 = \left(x^*, y^*\right) = \left(\dfrac{d\, e^{\delta\tau}}{\beta}, \dfrac{r}{\beta}\right)$, is a coexistence equilibrium which is biologically meaning full. We are only interested here in analyzing the biologically meaningful coexistence equilibrium.

The linearization of (1.1) about an equilibrium $\left(\overline{x}, \overline{y}\right)$ given by

$$\begin{bmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \end{bmatrix} = \begin{bmatrix} r - \beta\,\overline{y} & -\beta\,\overline{x} \\ 0 & -d \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \beta e^{-\delta\tau}\overline{y} & \beta e^{-\delta\tau}\overline{x} \end{bmatrix} \begin{bmatrix} u_1(t-\tau) \\ u_2(t-\tau) \end{bmatrix} \tag{1.2}$$

The associated characteristic equation is given by

$$\det\begin{bmatrix} r - \beta\,\overline{y} - \lambda & -\beta\,\overline{x} \\ \beta e^{-\delta\tau} e^{-\lambda\tau}\overline{y} & -d + \beta e^{-\delta\tau} e^{-\lambda\tau}\,\overline{x} - \lambda \end{bmatrix} = 0$$

$$\Rightarrow \left(r - \beta\,\overline{y} - \lambda\right)\left(-d + \beta e^{-\delta\tau} e^{-\lambda\tau}\,\overline{x} - \lambda\right) + \beta^2 \overline{x}\,\overline{y} e^{-\delta\tau} e^{-\lambda\tau} = 0$$

$$\Rightarrow \lambda^2 - \left(r - \beta\overline{y} - d + \beta e^{-\delta\tau} e^{-\lambda\tau}\overline{x}\right)\lambda + \beta^2 \overline{x}\,\overline{y} e^{-\delta\tau} e^{-\lambda\tau} + \left(r - \beta\overline{y}\right)\left(-d + \beta e^{-\delta\tau} e^{-\lambda\tau}\,\overline{x}\right) = 0$$

Now we define

$$F(\lambda) = \lambda^2 - \left(r - \beta\,\overline{y} - d + \beta e^{-\delta\tau} e^{-\lambda\tau}\overline{x}\right)\lambda + \beta^2 \overline{x}\,\overline{y} e^{-\delta\tau} e^{-\lambda\tau} + \left(r - \beta\,\overline{y}\right)\left(-d + \beta e^{-\delta\tau} e^{-\lambda\tau}\,\overline{x}\right) = 0 \tag{1.3}$$

At the equilibrium point $E_2 = \left(x^*, y^*\right) = \left(\dfrac{d\, e^{\delta\tau}}{\beta}, \dfrac{r}{\beta}\right)$, (1.3) becomes

$$F(\lambda) = \lambda^2 - \left(r - \beta\frac{r}{\beta} - d + \beta e^{-\delta\tau} e^{-\lambda\tau}\,\frac{d\, e^{\delta\tau}}{\beta}\right)\lambda + \beta^2\,\frac{d\, e^{\delta\tau}}{\beta}\,\frac{r}{\beta}\; e^{-\delta\tau} e^{-\lambda\tau} +$$
$$\left(r - \beta\frac{r}{\beta}\right)\left(-d + \beta\, e^{-\delta\tau} e^{-\lambda\tau}\,\frac{d\, e^{\delta\tau}}{\beta}\right) = 0$$

$$\Rightarrow F(\lambda) = \lambda^2 + \left(d - d e^{-\lambda\tau}\right)\lambda + r d e^{-\lambda\tau} = 0 \tag{1.4}$$

If $\tau = 0$, then (1.4) becomes

$$\lambda^2 + \left(d - d\right)\lambda + rd = 0$$

$$\Rightarrow \lambda^2 + rd = 0$$

$$\Rightarrow \lambda = \pm i\sqrt{r\,d} = \pm i\beta_0$$

Where

$$\beta_0 = \sqrt{rd} > 0$$

When $\tau = 0$, there are no real roots and two purely imaginary roots. Therefore, it is a centre. Now we can examine whether Hopf bifurcation will occur or not.

Consider,

$$G(r\,,\,\lambda) = \lambda^2 + rd \tag{1.5}$$

$$\frac{\partial G}{\partial \lambda} = 2\lambda$$

$$\frac{\partial G}{\partial \lambda}\Big|_{\lambda=\pm i\beta_0} = \pm 2i\beta_0 = \pm 2i\sqrt{rd} \neq 0$$

$$\frac{d\lambda}{dr} = -\frac{\partial G}{\partial r}/\frac{\partial G}{\partial \lambda} = -\frac{d}{2\lambda}$$

$$\therefore \frac{d\lambda}{dr}\Bigg|_{\lambda=\pm i\beta_0} = \mp\frac{d}{2i\beta_0} = 0 \pm \frac{i\,d}{2\sqrt{r\,d}}$$

$$\therefore \mathrm{Re}\frac{d\lambda}{dr}\Big|_{\lambda=\pm i\beta_0} = 0$$

The transversality condition does not satisfy. Therefore, if $\tau = 0$ Hopf bifurcation does not hold.

In case of positive delay, i.e. $\tau > 0$ the characteristic equation for the lineralized equation around the point $E_2 = \left(x^*, y^*\right)$ is given by

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0 \tag{1.6}$$

Where

$$P(\lambda) = \lambda^2 + d\lambda$$

$$Q(\lambda) = -d\,\lambda + r\,d$$

If $\tau > 0$, Let $\lambda = i\,\omega, \quad \omega > 0$ be a purely imaginary root of (1.6).

Now substituting $\lambda = i\,\omega$ in Eq. (1.6)

$$F(i\,\omega) = (i\,\omega)^2 + \left(d - d\,e^{-i\omega\tau}\right)i\,\omega + r\,d\,e^{-i\omega\tau} = 0$$

$$\Rightarrow -\omega^2 + i\,\omega d - i\,\omega d(\cos\omega\tau - i\sin\omega\tau) + r\,d\,(\cos\omega\tau - i\sin\omega\tau) = 0$$

$$\Rightarrow \left(-\omega^2 - \omega d \sin\omega\tau + r\,d \cos\omega\tau\right) + i\left(\omega d - \omega d \cos\omega\tau - r\,d \sin\omega\tau\right) = 0$$

Now separating the real and imaginary parts we obtain the system of transcendental equations

$$R(\omega) = -\omega^2 - \omega d \sin \omega\tau + rd \cos\omega\tau = 0 \tag{1.7}$$

$$S(\omega) = \omega d - \omega d \cos\omega\tau - rd \sin\omega\tau = 0 \tag{1.8}$$

Squaring and adding (1.7) and (1.8) we get,

$$\omega^2 d^2 + r^2 d^2 = \omega^4 + \omega^2 d^2$$

$$\Rightarrow \left(\omega^2 + r\,d\right)\left(\omega^2 - rd\right) = 0$$

But

$$\left(\omega^2 + r\,d\right) \neq 0, \text{ as } r > 0, d > 0, \omega > 0 .$$

$$\Rightarrow \omega^2 - rd = 0$$

$$\Rightarrow \omega = \omega_0 = \sqrt{r\,d} \text{ , as } \omega > 0 .$$

Therefore we have a positive $\omega = \omega_0 > 0$ such that equation (1.6) has purely imaginary roots. Eliminating $\sin(\omega\tau)$ from (1.7) and (1.8) , we get

$$\cos(\omega\tau) = \frac{\omega^2 r + \omega^2 d}{\omega^2 d + r^2 d}$$

Then τ_0 corresponding to ω_0 is given by

$$\tau_0 = \frac{1}{\omega_0} \arccos \frac{\omega_0^2 (r + d)}{\left(\omega_0^2 + r^2\right) d}$$

Hopf- bifurcation

We will now show that

$$\left[\frac{d(\mathrm{Re}\,\lambda)}{d\tau}\right]_{\tau=\tau_0} > 0$$

Note that all roots of (1.6) depend continuously on τ (see Busenberg & cooke, 1993), and as τ increase, a root of (1.6) may enter the right half plane only by crossing the imaginary axis(see. Beretta & Kuang 2002). Thus as $\tau > 0$ increases, roots of (1.6) may cross the imaginary axis only through a pair of non zero purely imaginary roots. To see if there is any stability switch as τ crosses τ_0, we take the help of some results by Cooke and Van den Driessche in Theorem 1 of [10]. We first look for purely imaginary roots of $\lambda = i\omega_0$, $\omega_0 > 0$ of (1.6). Equation (1.6) implies

$$|P(i\omega_0)| = |Q(i\omega_0)|$$

and this determines a set of possible values of ω_0. Our aim is to determine the direction of motion of λ as τ is varied. That is, we determine

$$\Theta = sign\left[\frac{d(\mathrm{Re}\,\lambda)}{d\tau}\right]_{\lambda=i\omega_0} = sign\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\omega_0}$$

Now, differentiating (1.6) with respect to τ, we get

$$\left[(2\lambda+d) - de^{-\lambda\tau} - \tau e^{-\lambda\tau}(rd - d\lambda)\right]\frac{d\lambda}{d\tau} = \lambda(rd - d\lambda)e^{-\lambda\tau}$$

$$\Rightarrow \left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(2\lambda+d)}{\lambda(rd-d\lambda)e^{-\lambda\tau}} - \frac{d}{\lambda(rd-d\lambda)} - \frac{\tau}{\lambda}$$

$$= \frac{(2\lambda+d)}{-\lambda(\lambda^2+d\lambda)} - \frac{d}{\lambda(rd-d\lambda)} - \frac{\tau}{\lambda}$$

Therefore

$$\Theta = sign\left[\mathrm{Re}\left(\frac{2\lambda+d}{-\lambda(\lambda^2+d\lambda)} - \frac{d}{\lambda(rd-d\lambda)} - \frac{\tau}{\lambda}\right)\right]_{\lambda=i\omega_0}$$

$$= sign\left[\mathrm{Re}\left(\frac{2i\omega_0+d}{-i\omega_0(i^2\omega_0^2+di\omega_0)} - \frac{d}{i\omega_0(rd-i\omega_0 d)} - \frac{\tau}{i\omega_0}\right)\right]$$

$$= sign\left[\mathrm{Re}\left(\frac{2i\omega_0+d}{d\omega_0^2+i\omega_0^3} - \frac{d}{d\omega_0^2+ird\omega_0} - \frac{\tau}{i\omega_0}\right)\right]$$

$$= sign\left[\mathrm{Re}\left(\frac{(2i\omega_0+d)(d\omega_0^2-i\omega_0^3)}{(d\omega_0^2)^2+\omega_0^6} - \frac{d(d\omega_0^2-ird\omega_0)}{d^2\omega_0^4+r^2d^2\omega_0^2} + \frac{i\tau}{\omega_0}\right)\right]$$

$$= sign\left[\frac{2\omega_0^4+d^2\omega_0^2}{d^2\omega_0^4+\omega_0^6} - \frac{d^2\omega_0^2}{d^2\omega_0^4+r^2d^2\omega_0^2}\right]$$

$$= sign\left[\frac{2\omega_0^2+d^2}{d^2\omega_0^2+\omega_0^4} - \frac{1}{\omega_0^2+r^2}\right]$$

$$= sign\left[\frac{2\omega_0^4 + 2\omega_0^2 r^2 + \omega_0^2 d^2 + r^2 d^2 - \omega_0^2 d^2 - \omega_0^4}{(d^2\omega_0^2 + \omega_0^4)\,(\omega_0^2 + r^2)}\right]$$

$$= sign\left[\frac{\omega_0^4 + 2\omega_0^2 r^2 + r^2 d^2}{(d^2\omega_0^2 + \omega_0^4)\,(\omega_0^2 + r^2)}\right] > 0$$

We have

$$\left[\frac{d(\operatorname{Re}\lambda)}{d\tau}\right]_{\omega=\omega_0\,,\,\tau=\tau_0} > 0$$

Therefore, the transversality condition holds and hence Hopf-bifurcation occurs at $\omega = \omega_0\,, \tau = \tau_0$ .So as τ increases i.e. $\tau \geq \tau_0$, a periodic solution will occur which is the case of Hopf-bifurcation. Hence if $\tau = 0$, there is a pair of purely imaginary roots and its represent centre. When τ increases to τ_0 i.e. $\tau \in (0, \tau_0)$, there is another pair of purely imaginary roots.

Section-2

Now we consider the logistic growth function of the Prey and Holling type I , functional response for predator on prey population. The model (1) becomes

$$\frac{dx\,(t)}{dt} = r\,x(t)\left(1 - \frac{x(t)}{K}\right) - \beta\;x(t)\;y(t)$$

$$\frac{dy(t)}{dt} = \beta e^{-\delta\tau}\;x(t-\tau)\;y(t-\tau) - d\;y(t) \qquad (2.1)$$

Positivity of solutions

Positivity of the system (2.1) can be easily proved as like as we proved for system (1.1).

Equilibria and Stability Analysis

The Model (2.1) has two equilibrium points: $\overline{E}_1 = (0\,,0)$ is a trivial equilibrium which is biologically meaningless and $\overline{E}_2 = \left(\frac{d\,e^{\delta\tau}}{\beta}, \frac{r}{\beta}\left(1 - \frac{d}{\beta K}e^{\delta\tau}\right)\right)$, is a coexistence equilibrium which would be biologically meaningful iff $\frac{r}{\beta}\left(1 - \frac{d}{\beta K}e^{\delta\tau}\right) > 0$ i.e. $\tau < \frac{1}{\delta}\ln\left(\frac{\beta K}{d}\right)$. For existence, this equilibrium point we need to be assumed $\frac{\beta K}{d} > 1$

. We are only interested here in analyzing the biologically meaningful coexistence equilibrium.

The linearization of (2.1) about an equilibrium $(\bar{x},\bar{y})$ given by

$$\begin{bmatrix}\dot{\bar{u}}_1(t)\\ \dot{\bar{u}}_2(t)\end{bmatrix}=\begin{bmatrix} r(1-\frac{2\bar{x}}{K})-\beta\bar{y} & -\beta\bar{x}\\ 0 & -d\end{bmatrix}\begin{bmatrix}\bar{u}_1(t)\\ \bar{u}_2(t)\end{bmatrix}+\begin{bmatrix}0 & 0\\ \beta e^{-\delta\tau}\bar{y} & \beta e^{-\delta\tau}\bar{x}\end{bmatrix}\begin{bmatrix}\bar{u}_1(t-\tau)\\ \bar{u}_2(t-\tau)\end{bmatrix} \tag{2.2}$$

The associated characteristic equation is given by

$$\det\begin{bmatrix} r(1-\frac{2\bar{x}}{K})-\beta\bar{y}-\lambda & -\beta\bar{x}\\ \beta e^{-\delta\tau}e^{-\lambda\tau}\bar{y} & -d+\beta e^{-\delta\tau}e^{-\lambda\tau}\bar{x}-\lambda\end{bmatrix}=0$$

$$\Rightarrow\left(r-\frac{2\bar{x}r}{K}-\beta\bar{y}-\lambda\right)\left(-d+\beta e^{-\delta\tau}e^{-\lambda\tau}\bar{x}-\lambda\right)+\beta^2\bar{x}\bar{y}e^{-\delta\tau}e^{-\lambda\tau}=0$$

$$\Rightarrow\lambda^2-\left(r-\frac{2\bar{x}r}{K}-\beta\bar{y}-d+\beta e^{-\delta\tau}e^{-\lambda\tau}\bar{x}\right)\lambda+\beta^2\bar{x}\bar{y}e^{-\delta\tau}e^{-\lambda\tau}+\left(r-\frac{2\bar{x}r}{K}-\beta\bar{y}\right)\left(-d+\beta e^{-\delta\tau}e^{-\lambda\tau}\bar{x}\right)=0$$

Now we define

$$\bar{F}(\lambda)=\lambda^2-\left(r-\frac{2\bar{x}r}{K}-\beta\bar{y}-d+\beta e^{-\delta\tau}e^{-\lambda\tau}\bar{x}\right)\lambda+\beta^2\bar{x}\bar{y}e^{-\delta\tau}e^{-\lambda\tau}+\left(r-\frac{2\bar{x}r}{K}-\beta\bar{y}\right)\left(-d+\beta e^{-\delta\tau}e^{-\lambda\tau}\bar{x}\right)=0 \tag{2.3}$$

At the equilibrium point $\bar{E}_2=(x^*,y^*)=\left(\frac{d\,e^{\delta\tau}}{\beta},\frac{r}{\beta}\left(1-\frac{d}{\beta K}e^{\delta\tau}\right)\right)$, (2.3) becomes

$$\bar{F}(\lambda)=\lambda^2-\left(r-\frac{2rd}{K\beta}e^{\delta\tau}-\beta\frac{r}{\beta}\left(1-\frac{d}{\beta K}e^{\delta\tau}\right)-d+\beta e^{-\delta\tau}e^{-\lambda\tau}\frac{de^{\delta\tau}}{\beta}\right)\lambda+\beta^2\frac{de^{\delta\tau}}{\beta}\frac{r}{\beta}\left(1-\frac{d}{\beta K}e^{\delta\tau}\right)e^{-\delta\tau}e^{-\lambda\tau}$$
$$+\left(r-\frac{2rd}{K\beta}e^{\delta\tau}-\beta\frac{r}{\beta}\left(1-\frac{d}{\beta K}e^{\delta\tau}\right)\right)\left(-d+\beta e^{-\delta\tau}e^{-\lambda\tau}\frac{de^{\delta\tau}}{\beta}\right)=0$$

$$\Rightarrow\lambda^2-\left(-d+d\,e^{-\lambda\tau}-\frac{rd}{K\beta}e^{\delta\tau}\right)\lambda+rd\,e^{-\lambda\tau}-\frac{rd^2}{\beta K}e^{\delta\tau}e^{-\lambda\tau}+\frac{rd^2}{\beta K}e^{\delta\tau}-\frac{rd^2}{\beta K}e^{\delta\tau}e^{-\lambda\tau}=0$$

$$\therefore\ \bar{F}(\lambda)=\lambda^2-\left(d\,e^{-\lambda\tau}-d-\frac{rd}{K\beta}e^{\delta\tau}\right)\lambda+\frac{rd^2}{\beta K}e^{\delta\tau}+\left(rd-\frac{2rd^2}{\beta K}e^{\delta\tau}\right)e^{-\lambda\tau}=0 \tag{2.4}$$

If $\tau=0$, then (2.4) becomes

$$\lambda^2-\left(-\frac{rd}{K\beta}\right)\lambda+\frac{rd^2}{\beta K}+\left(rd-\frac{2rd^2}{\beta K}\right)=0$$

$$\Rightarrow \lambda^2 + \frac{r\,d}{\beta K}\lambda + rd - \frac{r\,d^2}{\beta K} = 0$$

$$\therefore\ \lambda = -\frac{1}{2}\frac{r\,d}{\beta\,K} \pm \frac{1}{2}\sqrt{\left(\frac{r\,d}{\beta K}\right)^2 - 4\left(rd - \frac{rd^2}{\beta K}\right)}$$

We consider

$$\lambda_1 = -\frac{1}{2}\frac{r\,d}{\beta\,K} - \frac{1}{2}\sqrt{\left(\frac{r\,d}{\beta K}\right)^2 - 4rd\left(1 - \frac{d}{\beta K}\right)} \quad \text{and}$$

$$\lambda_2 = -\frac{1}{2}\frac{r\,d}{\beta\,K} + \frac{1}{2}\sqrt{\left(\frac{r\,d}{\beta K}\right)^2 - 4rd\left(1 - \frac{d}{\beta K}\right)}$$

Here the real part of λ_1 is negative, therefore the stability depends on another eigen value λ_2.

Theorem 2.1: If $\tau = 0$, then $\overline{E}_2$ is locally asymptotically stable if $d < \beta K$ and $\overline{E}_2$ does not exists if $d > \beta K$.

In case of positive delay i.e. $\tau > 0$, the characteristic equation for the lineralized equation around the point $\overline{E}_2$ is given by

$$\overline{P}(\lambda) + \overline{Q}(\lambda)e^{-\lambda\tau} = 0 \tag{2.5}$$

Where

$$\overline{P}(\lambda) = \lambda^2 + p_1\lambda + p_2$$

$$\overline{Q}(\lambda) = q_1\lambda + q_2$$

Here

$$p_1 = d + \frac{rd}{K\beta}e^{\delta\tau} > 0$$

$$p_2 = \frac{rd^2}{K\beta}e^{\delta\tau} > 0$$

$$q_1 = -d < 0$$

$$q_2 = rd - \frac{2rd^2}{\beta K}$$

If $\tau > 0$, Let $\lambda = i\,\omega, \quad \omega > 0$ be a purely imaginary root of (2.5).

Now substituting $\lambda = i\,\omega$ in Eq. (2.5)

$$\overline{F}(i\,\omega) = (i\,\omega)^2 - i\,\omega\left(d\,e^{-i\omega\tau} - d - \frac{rd}{K\beta}e^{\delta\tau}\right) + \frac{rd^2}{K\beta}e^{\delta\tau} + \left(rd - \frac{2rd^2}{K\,\beta}\right)e^{\delta\tau}e^{-i\omega\tau} = 0$$

$$\Rightarrow -\omega^2 - i\,\omega d(\cos\omega\tau - i\sin\omega\tau) + i\omega d + \frac{i\omega rd}{\beta K}e^{\delta\tau} + \frac{rd^2}{K\beta}e^{\delta\tau} + \left(rd - \frac{2rd^2}{K\beta}\right)e^{\delta\tau}(\cos\omega\tau - i\sin\omega\tau) = 0$$

$$\Rightarrow\left(-\omega^2 - \omega d\sin\omega\tau + \frac{rd^2}{\beta K}e^{\delta\tau} + rd\cos\omega\tau - \frac{2rd^2}{\beta K}e^{\delta\tau}\cos\omega\tau\right) +$$

$$i\left(-\omega d\cos\omega\tau + \omega d + \frac{\omega rd}{K\beta}e^{\delta\tau} - rd\sin\omega\tau + \frac{2rd^2}{K\beta}e^{\delta\tau}\sin\omega\tau\right) = 0$$

Now separating the real and imaginary parts we obtain the system of transcendental equations

$$\left(rd - \frac{2rd^2}{K\beta}e^{\delta\tau}\right)\cos\omega\tau - \omega d\sin\omega\tau = \omega^2 - \frac{rd^2}{\beta K}e^{\delta\tau} \tag{2.6}$$

$$\left(rd - \frac{2rd^2}{K\beta}e^{\delta\tau}\right)\sin\omega\tau + \omega d\cos\omega\tau = \omega d + \frac{\omega rd}{\beta K}e^{\delta\tau} \tag{2.7}$$

Squaring and adding (2.6) and (2.7) we get,

$$\left(rd - \frac{2rd^2}{\beta K}e^{\delta\tau}\right)^2 + \omega^2 d^2 = \left(\omega^2 - \frac{rd^2}{\beta K}e^{\delta\tau}\right)^2 + \left(\omega d + \frac{\omega rd}{\beta K}e^{\delta\tau}\right)^2$$

$$\Rightarrow\left(rd - \frac{2rd^2}{\beta K}e^{\delta\tau}\right)^2 + \omega^2 d^2 = \omega^4 - \frac{2\omega^2 rd^2}{\beta K}e^{\delta\tau} + \frac{r^2d^4}{\beta^2K^2}e^{2\delta\tau} + \omega^2\left(d + \frac{rd}{K\beta}e^{\delta\tau}\right)^2$$

$$\Rightarrow \omega^4 - \left(\frac{2rd^2}{\beta K}e^{\delta\tau} - (d + \frac{rd}{\beta K}e^{\delta\tau})^2 + d^2\right)\omega^2 + \frac{r^2d^4}{\beta^2K^2}e^{2\delta\tau} - \left(rd - \frac{2rd^2}{\beta K}e^{\delta\tau}\right)^2 = 0$$

$$\Rightarrow \omega^4 - \left(\frac{2rd^2}{\beta K}e^{\delta\tau} - d^2 - \frac{2rd^2}{K\beta}e^{\delta\tau} - \frac{r^2d^2}{K^2\beta^2}e^{2\delta\tau} + d^2\right)\omega^2 + \frac{r^2d^4}{\beta^2K^2}e^{2\delta\tau} - r^2d^2 + \frac{2r^2d^3}{\beta K}e^{\delta\tau} - \frac{4r^2d^4}{\beta^2K^2}e^{2\delta\tau} = 0$$

$$\Rightarrow \omega^4 + \frac{r^2d^2}{\beta^2K^2}e^{2\delta\tau}\omega^2 + \frac{2r^2d^3}{\beta K}e^{\delta\tau} - r^2d^2 - \frac{3r^2d^4}{\beta^2K^2}e^{2\delta\tau} = 0$$

$$\therefore \omega^2 = -\frac{1r^2d^2}{2K^2\beta^2}e^{2\delta\tau} \pm \frac{1}{2}\sqrt{\left(\frac{r^2d^2}{K^2\beta^2}e^{2\delta\tau}\right)^2 - 4\left(\frac{2r^2d^3}{\beta K}e^{\delta\tau} - r^2d^2 - \frac{3r^2d^4}{\beta^2K^2}e^{2\delta\tau}\right)}$$

Since $\omega^2 > 0$

$$\therefore -\frac{1r^2d^2}{2K^2\beta^2}e^{2\delta\tau} + \frac{1}{2}\sqrt{\left(\frac{r^2d^2}{K^2\beta^2}e^{2\delta\tau}\right)^2 - 4\left(\frac{2r^2d^3}{\beta K}e^{\delta\tau} - r^2d^2 - \frac{3r^2d^4}{\beta^2K^2}e^{2\delta\tau}\right)} > 0$$

$$\therefore -\frac{1r^2d^2}{2K^2\beta^2}e^{2\delta\tau} > -\frac{1}{2}\sqrt{\left(\frac{r^2d^2}{K^2\beta^2}e^{2\delta\tau}\right)^2 - 4\left(\frac{2r^2d^3}{\beta K}e^{\delta\tau} - r^2d^2 - \frac{3r^2d^4}{\beta^2K^2}e^{2\delta\tau}\right)}$$

$$\Rightarrow \left(\frac{r^2d^2}{K^2\beta^2}e^{2\delta\tau}\right)^2 < \left(\frac{r^2d^2}{K^2\beta^2}e^{2\delta\tau}\right)^2 - 4\left(\frac{2r^2d^3}{\beta K}e^{\delta\tau} - r^2d^2 - \frac{3r^2d^4}{\beta^2K^2}e^{2\delta\tau}\right)$$

$$\Rightarrow 0 < -\left(\frac{2r^2d^3}{\beta K}e^{\delta\tau} - r^2d^2 - \frac{3r^2d^4}{\beta^2K^2}e^{2\delta\tau}\right)$$

$$\Rightarrow \frac{2r^2d^3}{\beta K}e^{\delta\tau} < r^2d^2 + \frac{3r^2d^4}{\beta^2K^2}e^{2\delta\tau}$$

$$\Rightarrow \frac{3d^2}{\beta^2K^2}e^{2\delta\tau} - \frac{2d}{\beta K}e^{\delta\tau} + 1 > 0$$

$$\Rightarrow e^{2\delta\tau} - \frac{2\beta K}{3d}e^{\delta\tau} + \frac{\beta^2K^2}{3d^2} > 0$$

$$\Rightarrow \left(e^{\delta\tau} - \frac{\beta K}{3d}\right)^2 + \left(\frac{\sqrt{2}\,\beta K}{3d}\right)^2 > 0 \tag{2.8}$$

This expression (2.8) is always true as all the parameters here are positive; therefore we have a positive $\omega = \omega_0 > 0$ such that equation (2.5) has a purely imaginary root. We can also find the value of τ_0 corresponding to ω_0 in a similar fashion that we have already discussed in section 1.

Hopf- bifurcation

We will now show that

$$\left[\frac{d(\mathrm{Re}\lambda)}{d\tau}\right]_{\tau=\tau_0} > 0$$

This will signify that there exists at least one eigenvalue with positive real part for

$\tau > \tau_0$. Also, the conditions for Hopf bifurcation [11] are then satisfied yielding the required periodic solution. To see if there is any stability switch as τ crosses τ_0, we take the help of some results by Cooke and Van den Driessche in Theorem 1 of [10]. We first look for purely imaginary roots of $\lambda = i\omega_0$ of (2.5). Equation (2.5) implies

$$|P(i\omega_0)| = |Q(i\omega_0)|$$

and this determines a set of possible values of ω_0. Our aim is to determine the direction of motion of λ as τ is varied. That is, we determine

$$\Theta = sign\left[\frac{d(\mathrm{Re}\,\lambda)}{d\tau}\right]_{\lambda=i\omega_0} = sign\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\omega_0}$$

Now, differentiating (2.5) with respect to τ, we get

$$\left[(2\lambda + p_1) + e^{-\lambda\tau}q_1 - \tau e^{-\lambda\tau}(q_1\lambda + q_2)\right]\frac{d\lambda}{d\tau} = \lambda(q_1\lambda + q_2)e^{-\lambda\tau}$$

$$\Rightarrow \left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(2\lambda + p_1)}{\lambda(q_1\lambda + q_2)e^{-\lambda\tau}} + \frac{q_1 e^{-\lambda\tau}}{\lambda(q_1\lambda + q_2)e^{-\lambda\tau}} - \frac{\tau}{\lambda}$$

$$= \frac{(2\lambda + p_1)}{-\lambda(\lambda^2 + p_1\lambda + p_2)} + \frac{q_1}{\lambda(q_1\lambda + q_2)} - \frac{\tau}{\lambda}$$

Therefore

$$\Theta = sign\left[\mathrm{Re}\left(\frac{2\lambda + p_1}{-\lambda(\lambda^2 + p_1\lambda + p_2)} + \frac{q_1}{\lambda(q_1\lambda + q_2)} - \frac{\tau}{\lambda}\right)\right]_{\lambda=i\omega_0}$$

$$= sign\left[\mathrm{Re}\left(\frac{2i\omega_0 + p_1}{-i\omega_0(i^2\omega_0^2 + p_1 i\omega_0 + p_2)} + \frac{q_1}{i\omega_0(q_1 i\omega_0 + q_2)} - \frac{\tau}{i\omega_0}\right)\right]$$

$$= sign\left[\mathrm{Re}\left(\frac{2i\omega_0 + p_1}{p_1\omega_0^2 + i(\omega_0^3 - p_2\omega_0)} + \frac{q_1}{-q_1\omega_0^2 + iq_2\omega_0} - \frac{\tau}{i\omega_0}\right)\right]$$

$$= sign\left[\mathrm{Re}\left(\frac{(2i\omega_0 + p_1)(p_1\omega_0^2 - i\omega_0^3 + ip_2\omega_0)}{(p_1\omega_0^2)^2 + (\omega_0^3 - p_2\omega_0)^2} + \frac{q_1(-q_1\omega_0^2 - q_2 i\omega_0)}{q_1^2\omega_0^4 + q_2^2\omega_0^2} + \frac{i\tau}{\omega_0}\right)\right]$$

$$= sign\left[\frac{2\omega_0(\omega_0^3 - p_2\omega_0) + p_1^2\omega_0^2}{p_1^2\omega_0^4 + (\omega_0^3 - p_2\omega_0)^2} - \frac{q_1^2\omega_0^2}{q_1^2\omega_0^4 + q_2^2\omega_0^2}\right]$$

$$= sign\left[\frac{\omega_0^6 q_1^2 + 2\omega_0^4 q_2^2 - 2p_2 q_2^2 \omega_0^2 + p_1^2 q_2^2 \omega_0^2 - q_1^2 p_2^2 \omega_0^2}{(p_1^2 \omega_0^4 + \left(\omega_0^3 - p_2\omega_0\right)^2)\left(q_1^2\omega_0^4 + q_2^2\omega_0^2\right)}\right]$$

$$= sign\left[\frac{\omega_0^6 q_1^2 + 2\omega_0^2 q_2^2(\omega_0^2 - p_2) + \omega_0^2(p_1^2 q_2^2 - q_1^2 p_2^2)}{(p_1^2 \omega_0^4 + \left(\omega_0^3 - p_2\omega_0\right)^2)\left(q_1^2\omega_0^4 + q_2^2\omega_0^2\right)}\right]$$

Therefore, the transversality condition will hold and hence Hopf-bifurcation will occur at $\omega = \omega_0, \tau = \tau_0$. i.e.

$\left[\frac{d(\mathrm{Re}\lambda)}{d\tau}\right]_{\omega=\omega_0,\tau=\tau_0} > 0$, if and only if

$$\omega_0^2 - p_2 > 0 \Rightarrow \omega_0^2 > \frac{rd^2}{\beta K} e^{\delta\tau} \tag{2.9}$$

and

$$p_1^2 q_2^2 - q_1^2 p_2^2 > 0$$

$$\Rightarrow \left(\frac{rd}{\beta K} e^{\delta\tau}\right)^2 \left(rd - \frac{2rd^2}{\beta K}\right)^2 - d^2\left(\frac{rd^2}{\beta K} e^{\delta\tau}\right)^2 > 0$$

$$\Rightarrow \left(\frac{rd}{\beta K} e^{\delta\tau}\right)^2 \left[\left(rd - \frac{2rd^2}{\beta K}\right)^2 - d^4\right] > 0$$

$$\Rightarrow r^2\left(1 - \frac{2d}{\beta K}\right)^2 > d^2 \tag{2.10}$$

Result and discussion

It is broadly well known that past history as well as current conditions can influence population dynamics and such interactions has motivated the introduction of time delays in population growth models. In most of the natural systems, population of one species does not respond instantaneously to changes in the environment or the interactions with other species of populations within the community. It is believed that the time delays have a destabilizing effect in the models of population dynamics and often time delays are responsible for the population oscillations in constant environment. Discrete time delay has ability to alter the dynamical behavior of a model system significantly. In this paper, a mathematical model has been proposed and analyzed to study the dynamics of a predator-prey system due to the time lags for the conversion of biomass and considering different growth functions of prey. The model has been analyzed in two cases: first when growth function of prey population is monotonic and second when growth function of prey population is logistic. In this paper, we have also made an attempt to understand the effect of gestation delay on dynamical behavior of a prey–predator system. Gestation delay is the time interval between the moments when an individual prey is killed and when the corresponding biomass is added to the predator population. As the growth rate of predator species solely depends upon the amount of biomass added (in predator's population density) due to the prey killing, the presence of gestation delay in predator's growth affect the abundance of predators, as there are some possibilities of predator's death during this gestation period before going to reproduction or growth. Linear stability analysis reveals the fact that for the monotonic growth rate of prey, in the absence of delay the coexistence equilibrium is a centre. But for the logistic growth function of prey, it is locally asymptotically stable if $d < \beta K$ and it does not exist if $d > \beta K$. Biologically it implies that for maintaining coexistence between the predator- prey interacting populations, balance growth rate of prey and the carrying capacity of an environment is a crucial matter. Oscillation in population density is quite natural and commonly observed in most of the prey–predator based ecosystems. We observed that for the monotonic growth rate of prey, in absence of delay, Hopf bifurcation is not possible but in case of positive delay Hopf bifurcation is possible without any condition and there is a periodic solution which is the case of Hopf bifurcation. Biologically it implies that gestation delay is crucial for a predator-prey interacting system. In case of the logistic growth of prey, Hopf bifurcation is possible under the conditions

$$\omega_0^2 > \frac{rd^2}{\beta K} e^{\delta\tau} \text{ and } r^2\left(1 - \frac{2d}{\beta K}\right)^2 > d^2 .$$

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