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p^{th} order of Entire Harmonic Function

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Abstract:

In this paper we are reporting the p^{th} lower order and p^{th} lower type of an entire function H(r,θ,ϕ). These function are obtained by various characterization in terms of (α_n) defined in [1] we also defined coefficient characterization of order and type of H(r,θ,ϕ)

Keywords: Entire harmonic function; order; type

1. Introduction:

If H(r,θ,ϕ) is a function and is harmonic in a neighborhood of origin in R^3.

H (r,θ,ϕ) has fallowing expansion in spherical co-ordinate

$$H(r,\theta,\phi)=\sum_{n=o}^{\infty}\sum_{m=o}^{\infty}\{a_{nm}^{(1)}\cos m\phi + a_{nm}^{(2)}\sin n\phi\}r^n p_n^m(\cos\theta) \tag{1. 1}$$

Where $a_{mn}^{(1)}$ & $a_{mn}^{(2)}$ are different coefficients.

This series converges absolutely and uniformly on a compact set of largest open ball centered at the origin which omits singularities of H(r,θ,ϕ).

Here x=rcos θ y=rcos θ cos ϕ z=rsin θ sin ϕ

$p_n^m(x)$ are associated Legendre function of first kind, n^{th} degree of order m function was defined as

$$p_n^m(x)=\left(1-x^2\right)^{\frac{m}{2}}\frac{d^m}{dx^m}(\,p_n^m(x)\,).$$

For H(r,θ,ϕ) entire define

$$M(r) = M(r, H) = \max_{\theta,\phi} H(r,\theta,\phi) \tag{1.2}$$

by [1] the pth order ρ^* and pth type T* of $H(r,\theta,\phi)$ are defined as

$$\rho^* = \lim_{r\to\infty}\sup \frac{\log^{[p]} M(r)}{\log r} \tag{1.3}$$

$$T^* = \lim_{r\to\infty}\sup \frac{\log^{[p-1]} M(r)}{r^{\rho^*}} \tag{1.4}$$

for p=1 the above definition same with classical definition of order and type.

Lower pth order λ^* and lower type τ^* are defined by [3] as

$$\lambda^* = \lambda^*(H) = \lim_{r\to\infty}\inf \frac{\log^{[p]} M(r)}{\log r} \tag{1.5}$$

$$\tau^* = \tau^*(H) = \lim_{r\to\infty}\inf \frac{\log^{[p-1]} M(r)}{r^{\rho^*}} \tag{1.6}$$

for p=2, 3, 4 ⋯ where $\log^{[0]}x = x$ and $\log^{[p]} = \log(\log^{[p-1]}x)$ we have consider pth lower order and pth lower type of harmonic function H (r,θ,ϕ) and obtain various characterization of these in terms of (α_n).

Defined by [1] as

$$\alpha_n = \max\left\{\frac{(n+m)!}{(m-n)!}\right\}^{1/2} |a_{mn}^i| \text{ i=1, 2.} \tag{1.7}$$

Also, by [1] defined function as

$$f(z) = \sum_{n=\infty}^{\infty} \alpha_n \left(1+n^{-1/2}\right)^n z^n \tag{1.8}$$

$$g(z) = \sum_{n=o}^{\infty} \alpha_n (1+2n)^{-1/2} z^n$$

where α_n defined above.

Lemma 1: If $H(r,\theta,\phi)$ is entire Harmonic function the f(z) and g(z) are also entire function of complex variable z further

$$2^{-1} m(r,g) \le M(r) \le 2M(r,f) \tag{1.9}$$

$$m(r,g) = \max_n \left\{\alpha_n (1+2n)^{-1}\right\}$$

$$M(r,f) = \max_{|z|\le r}\{|f(z)|\} \tag{1.10}$$

This result is obtained by Frayant [5, pp 27_28].

Lemma 2: Let f(z) and g(z) are entire functions defined as above then p^{th} order and p^{th} type of f(z) and g(z) are equal.

PROOF Let F(z)= $\sum_{n=o}^{\infty} a_n z^n$ be any entire function of p^{th} order ρ^*(F)and p^{th} type T^*(F)

Then it will be known by S. K. Bajpai G. P. Kapoor and O. P. Junenja [2]

$$\rho^*(F) = \lim_{n\to\infty} \sup \frac{n \log^{[p-1]} n}{\log(a_n)^{-1}} \tag{1. 11}$$

$$T^*(F) = \lim_{n\to\infty} \sup \frac{\log^{[p-2]} n}{e^{\lambda^*}}$$

Here for the function

$$\text{f(z)} = \sum_{n=\infty}^{\infty} \alpha_n \left(1 + n^{-1/2}\right)^n Z^n$$

we have

$$\frac{1}{\rho^*(f)} = \lim_{n\to\infty} \inf \frac{\log\left\{\alpha_n \left(1 + n^{-1/2}\right)^n\right\}^{-1}}{n \log^{[p-1]} n}$$

$$= \lim_{n\to\infty} \inf \frac{\log \alpha_n{}^{-1} - n \log\left(1 + n^{-1/2}\right)}{n \log^{[p-1]} n}$$

$$= \lim_{n\to\infty} \inf \frac{\log(\alpha_n)^{-1}}{n \log^{[p-1]} n}$$

Similarly for

$$\text{g(z)} = \sum_{n=o}^{\infty} \alpha_n (1 + 2n)^{-1/2} z^n$$

$$\frac{1}{\rho^*(g)} = \lim_{n\to\infty} \inf \frac{\log\left\{\alpha_n (1 + 2n)^{-1/2}\right\}^{-1}}{n \log^{[p-1]} n}$$

$$= \lim_{n\to\infty} \inf \frac{\log\left\{\alpha_n\right\}^{-1} + 1/2 \log(1 + 2n)}{n \log^{[p-1]} n}$$

$$= \lim_{n\to\infty} \inf \frac{\log\{\alpha_n\}^{-1}}{n \log^{[p-1]} n}$$

Hence $\rho^*(f) = \rho^*(g)$ since f and g are of same order using (2. 2) we get $T^*(f) = T^*(g)$.

2. Result and Discussions

Theorem -1 let $H(r,\theta,\phi)$ be an entire Harmonic function of pth order $\rho^* and$ pth lower order λ^* and pth type T*also lower pth type τ^* if f(z) and g(z) are entire functions defined above then

$$\rho^*(f)=\rho^*(g)=\rho^* \quad (2.1)$$

$$T^*(f)=T^*(g)=T^* \quad (2.2)$$

$$\lambda^*(g)\le\lambda^*\le\lambda^*(f) \quad (2.3)$$

$$\tau^*(g)\le\tau^*\le\tau^*(f) \quad (2.4).$$

Proof : By Srivastava's study[1]

$$2^{-1}m(r,g)\le M(r)\le 2M(r,g)$$

We have,

$$\lim_{r\to\infty}\sup\inf\frac{\log^p M(r,g)}{\log r}\le\lim_{r\to\infty}\sup\inf\frac{\log^{[p]} M(r)}{\log r}\le\lim_{r\to\infty}\sup\inf\frac{\log^{[p]} M(r,f)}{\log r}$$

$\log^{[p]} M(r,F)\cong\log^{[p]} m(r,f)$ as r→ ∞ result is defined in [6]

Hence from above we get

$$\rho^*(g)\le\rho*\le\rho^*(f)\ ;\ \lambda^*(g)\le\lambda*\le\lambda^*(f) \quad (2.\ 5)$$

since $\rho^*(g)=\rho^*(g)$

Thus we obtained (2. 1) and (2. 3) from (2. 5)

$$\lim_{r\to\infty}\sup\frac{\log^{[p-1]} m(r,g)}{r^{\rho^*}}\le\lim_{r\to\infty}\sup*\frac{\log^{[p-1]} M(r)}{r^{\rho^*}}\le\lim_{r\to\infty}\sup\frac{\log^{[p-1]} M(r,f)}{r^{\rho^*}}$$

Hence from lemma (2) we have (2. 2) and (2. 4).

Theorem 2: Let $H(r,\theta,\phi)$ be an entire function of order ρ^* and lower order λ^* and lower type τ^*. If $\left(\frac{\alpha_n}{\alpha_{n+1}}\right)$ is non decreasing function for n>n$_o$ then

$$\lambda^*=\lim_{r\to\infty}\sup\frac{n\log^{[p]} n}{\log\{\alpha_n\}^{-1}} \quad (2.\ 6)$$

Proof: For entire function

$F(z)=\sum_{n=o}^{\infty}a_n z^n$ $\left|\frac{a_n}{a_{n+1}}\right|$ non decreasing function of n for n>n$_o$ then

$\lambda^*(F)=\lim_{n\to\infty}\sup\frac{n\log^{[p]} n}{\log\{a_n\}^{-1}}$ If $\left(\frac{\alpha_n}{\alpha_{n+1}}\right)$ be a non decreasing function of n then

$$\lambda^*(f) = \lim_{n\to\infty} \frac{n\log^{[p]} n}{\log\{\alpha_n\}^{-1} - n\log\left(1+n^{-1/2}\right)}$$

$$= \lim_{n\to\infty}\inf \frac{n\log^{[p]} n}{\log\{\alpha_n\}^{-1}}$$

Similarly for $g(z) = \sum_{n=o}^{\infty} \alpha_n (1+2n)^{-1/2} z^n$

$$\lambda^*(g) = \lim_{n\to\infty}\inf \frac{n\log^{[p]} n}{\log\{\alpha_n\}^{-1} - 1/2\log(1+2n)} = \lim_{n\to\infty}\inf \left\{\frac{n\log^{[p]} n}{\log\{\alpha_n\}^{-1}}\right\}$$

From (2. 1) we have $\lambda^* = \lim_{n\to\infty} \frac{n\log^{[p]} n}{\log\{\alpha_n\}^{-1}}$.

Theorem 3: Let $H(r,\theta,\phi)$ be an entire Harmonic function lower p^th^ order λ^* and $\left(\frac{\alpha_n}{\alpha_{n+1}}\right)$ is non decreasing function for $n>n_o$ then

$$\lambda^* = \lim_{n\to\infty} \frac{\log^{[p]} n}{\log\left(\alpha_n / \alpha_{n+1}\right)} \tag{2. 7}$$

Proof: For an entire function

$$F(z) = \sum_{n=o}^{\infty} a_n z^n$$

defined by [2]

$$\lambda^*(F) = \lim_{n\to\infty}\inf \frac{\log^{[p]} n}{\log\left\{\frac{a_n}{a_{n+1}}\right\}}$$

Provided $\left|\frac{a_n}{a_{n+1}}\right|$ non decreasing function of n for $n < n_o$

Using the condition on $[\alpha_n]$ we can easily show as in above theorem

$$\lambda^*(f) = \lim_{n\to\infty}\inf \frac{\log^{[p]} n}{\log\left(\alpha_n / \alpha_{n+1}\right)}$$

Applying to $g(z) = \sum_{n=o}^{\infty} (1+2n)^{-1/2} \alpha_n z^n$

We have

$$\lambda^*(f) = \lim_{n\to\infty} \frac{\log^{[p]} n}{\log\left(\alpha_n / \alpha_{n+1}\right) + 1/2 \log\left(1+2(n+1) / 1+2n\right)}$$

$$= \lim_{n\to\infty} \frac{\log^{[p]} n}{\log\left(\alpha_n / \alpha_{n+1}\right)}$$

Thus we relation by using (2. 3)

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