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Approximation for the Final State of Generalized Epidemic Process

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Abstract

A generalized epidemic model is considered that describes the spread of an infectious disease of the SIR type with any specified distribution for the infectious period. The statistic under study is the number of susceptible who ultimately survive the disease. Daniels in a pioneering paper established for a particular case that when the population is large this variable may have a Poisson like behaviour. In this paper a necessary and sufficient condition is derived that guarantees the validity of such a Poisson approximation for the generalized epidemic.

Keywords: Reed-Frost, House holds, SPLT, PLT

1. Introduction

There is a considerable amount of mathematical literature concerned with the spread of an infectious disease of the susceptible –infected-removed (SIR) type.

Consider a population of n initial susceptible individuals and let the epidemic start by introducing m newly infected individuals. It is assumed that any pair of individuals makes contact at the points of a homogeneous Poisson process of rate β and that contact between different pairs is mutually independent. A susceptible if ever contacted by an infective is infected and becomes immediately infectious. Any infective i , initial or subsequent is infectious for a period of time of random length D_i . All the $D_i^{'}s$ are i.i.d and distribution of variables D. After that period, the infective dies or is immune and plays no further role in infectious process.

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This model is referred to here as the generalized epidemic which corresponds to the particular case where D is exponentially distributed of parameter μ . Now for this generalized epidemic, it is clear that the disease process eventually terminates at some finite time, A as soon as there are no more infectious present in the population. Then a statistic of great relevance is $S(A)$, the ultimate number of susceptible surviving the disease. When the interest is focussed on the variable $S(A)$, the generalized epidemic covers another standard model known as the Reed-Frost epidemic which is obtained by supposing that D is equal to some constant.

Moreover as far as $S(A)$ is concerned, the model can be reformulated as a special case of the randomized Reed-Frost epidemic. The latter model describes a similar SIR infection schema, but here it is supposed that during its infectious period, any infective i fails to transmit the disease to any given susceptible with random probability θ_i . All the $\theta_i^{'}s$ are i.i.d and distributed as the variable θ . Thus the generalized epidemic corresponds to the situation where $\theta = \exp(-\beta D)$.

The distribution of $S(A)$ is studied by Picard and Lefevre and they derived a compact expression for its probability generating function in terms of a non-standard family of polynomials. Much of these researches deal with the asymptotic behaviour of $S(A)$ as the initial susceptible population size is large $(n \to \infty)$ and the initial global infection rate per infective is constant $(\beta = \beta_n / n \text{ with } \beta_n \to \beta)$. Under these conditions there exists a threshold phenomenon such that for small outbreaks the final size $n - S(A)$ is finite and distributed as total progeny in a branching process, while for large outbreaks the final size is a positive fraction of n and $S(A)$ has a Gaussian limit approximation. The alternative asymptotic approximation of $S(A)$ by a Poisson law is given in this paper.

Our purpose is precisely to provide a thorough treatment of this problem of Poisson approximation for the final state of the generalized epidemic model. We construct a sequence of epidemics indexed by $n \to \infty$ and (defined as) from the parameters (n, m_n, β_n, D) . The distribution of D is supposed to be independent of n , which is not restrictive in practice. Let A_n be the end of the epidemic and let $S_n(\infty) \equiv S_n(A_n)$ be the ultimate number of susceptible with law denoted by $l(S_n(\infty))$. The main theorem gives a necessary and sufficient condition on D in order that for any sequence $\{m_n\}$ there exists a sequence $\{\beta_n\}$ such that the distributions $l(S_n(\infty))$ converge weakly to $P(b), 0 < b < \infty,$ as $n \to \infty$.

Our method, natural and powerful exploits directly the probabilistic structures of the epidemic process. It is based on two key ideas, namely, the building of an equivalent markovian representation of the model and the application of a suitable coupling via a random walk.

2. The Main Result

Consider a sequence of generalized epidemic models indexed by $n \to \infty$ and with parameters (n, m_n, β_n, D) . Let $S_n(\infty)$ be the final size of the susceptible population.

Definition 1

The quantity $S_n(\infty)$ is said to obey a Poisson limit theorem (PLT) if any sequence $\{m_n\}$, there exists a sequence $\{\beta_n\}$ such that

$$l\left(S_n(\infty)\right) \to {}_w P(b), 0 < b < \infty \text{ as } n \to \infty \qquad \to (1)$$

It is clear that if $m_n = 1, n \geq 1$ then for any n and β_n .

$$P\left(S_n(\infty) = n \middle| m_n = 1\right) \geq P(D = 0) \qquad \to (2)$$

Thus, a PLT implies that

$$P(D = 0) = 0 \qquad \to (3)$$

From now on, we will suppose that the condition (3) holds true.
Put

$$h(x) = \int_0^x P(D > y)dy \qquad \to (4)$$

$$g(x) = \int_x^\infty P(D > y)dy \qquad \to (5)$$

$$t(x) = 1 - E\left[e^{-xD}\right] \qquad \to (6)$$

Here after, we will use the concept of slowly varying function. An extensive treatment of this subject can be found in the book by Bingham, Goldic and Teugels (1987).

Proposition 1

The quantity $S_n(\infty)$ obeys a Poisson limit theorem (PLT) if and only if the two following conditions are satisfied.

$h(x)$ is a slowly varying function $\to (7)$

$$x \ln(x) P(D > x) / h(x) \to 0 \text{ as } n \to \infty \qquad \to (8)$$

Moreover under (7) and (8) a PLT holds with a given sequence of $\{\beta_n\}$, if and only if this sequence satisfies the condition.

$(n+m_n)t(\beta_n)-l_n(n/b_n)\to 0$ with $b_n \to b$ as $n\to\infty$, Or equivalently

$(n+m_n)\beta_n h\left(1/\beta_n\right)-\ln\left(n/\beta_n\right)\to 0$ with $b_n \to b$

Under (7) and (8)

$$x\,P\left(D>x\right)\to 0 \qquad \to (9)$$

$$h\left(x\right)/\ln\left(x\right)\to 0 \qquad \to (10)$$

$$\ln\left(x\right)\int_0^x P\left(D>t\right)dt/\left[xh\left(x\right)\right]\to 0 \text{ as } x\to\infty \qquad \to (11)$$

Proof

We start by showing that (8) alone implies (9) and (10).

Put $\varepsilon_x = x\ln\left(x\right)P\left(D>x\right)/h\left(x\right), x>0$. By (8), there exists x_0 such that $\varepsilon_{x_0}\le \ln\left(2\right)$ for all $x\ge x_0$. We have for $x\ge x_0$

$$h\left(2x\right)=h\left(x\right)+\int_x^{2x} P\left(D>t\right)dt \le h\left(x\right)+xP\left(D>x\right)$$

$$=h\left(x\right)\left[1+\varepsilon_x/\ln\left(x\right)\right]\le \ln\left(2x\right)\left[h\left(x\right)/\ln\left(x\right)\right]$$

By iterating, we obtain for $x\ge 2^{k-1}x_0, h\ge 1$ that

$$h\left(x\right)/\ln\left(x\right)\ge h\left(2x\right)/\ln\left(2x\right)\ge .....\ge h\left(2^k x\right)/\ln\left(2^k x\right) \qquad \to (12)$$

Since $h\left(x\right)$ is increasing, we deduce from (12) that for $x\in\left[2^{k-1}x_0, 2^k x_0\right], k\ge 1$

$$h\left(x\right)\le h\left(2^k x_0\right)\le \ln\left(2^k x_0\right)h\left(x_0\right)/\ln\left(x_0\right)\le \ln\left(2x\right)h\left(x_0\right)/\ln\left(x_0\right)\le c\left(x_0\right)\ln\left(x\right) \qquad \to (13)$$

for same constant $c\left(x_0\right)$. Inserting (13) in (8) then yields (9). Moreover fix $u>e$. We have for $x\ge u$,

$$h\left(x\right)\le u+\int_u^x P\left(D>t\right)dt$$

$$\le u+\left[\sup_{t\ge u}\left[P\left(D>t\right)t\right]\right]\ln\left(x\right) \qquad \to (14)$$

Choosing $u=\sqrt{\ln\left(x\right)}$ in (14) and using (9) then leads to (10). By (7) $h\left(x\right)/\ln\left(x\right)$ is a slowly varying function.

Thus we have

$$\int_e^x\left[h\left(t\right)/\ln\left(t\right)\right]dt\le \tilde{c}\left(h\right)x\,h\left(x\right)/\ln\left(x\right) \qquad \to (15)$$

For some $\tilde{c}\left(h\right)$. From (15) we find, for $x>e$

$$\int_0^x P(D>t)tdt=\int_0^{x^{1/3}} P(D>t)tdt+\int_{x^{1/3}}^x \varepsilon_t[h(t)/\ln(t)dt$$

$$\leq x^{2/3} + \varepsilon_{x^{1/3}} \int_{x^{1/3}}^{x} \left[h(t)/\ln(t) \right] dt$$

$$\leq x^{2/3} + \varepsilon_{x^{1/3}} \tilde{c}(h) x h(x)/\ln(x)$$

which yields

$$\ln(x) \int_0^x P(D > t) t dt \Big/ x h(x) \leq \ln(x) \Big/ x^{1/3} h(x) + \varepsilon_{x^{1/3}} \tilde{c}(h) \quad \to (16)$$

Since by (8) $\varepsilon_{x^{1/3}} \to 0$ as $x \to \infty$ (11) follows directly from (16).

Thus we have $h(x)/\ln(x) \to 0$ and $xP(D > x) \to 0$ as $x \to \infty$.

Corollary 1

Suppose that there are positive constants and x_0 such that

$$P(D > x) = c/(x \ln(x)) \text{ for all } x > x_0 \quad \to (17)$$

Then $S_n(\infty)$ obeys a PLT (with $E(D) = \infty$).

Proof

Under (17) we have for $x > x_0$,

$$h(x) = \int_0^x P(D > y) dy = C_1 \ln \ln(x) + C_2$$

C_1 and C_2 being appropriate constants. Thus $h(x)$ is a slowly varying function as required by (7). Moreover we directly obtain that (8) too is satisfied which yields the result.

For many applications $E(D)$ will be finite. If is easily seen that Proposition 1 then becomes the following corollary.

Corollary 2

When $E(D) < \infty, S_n(\infty)$ obeys a PLT if and only if

$$x \ln(x) P(D > x) \to 0 \text{ as } x \to \infty \quad \to (18)$$

Under (18) a PLT holds with any given sequence $\{\beta_n\}$ such that

$$(n + m_n)\beta_n \left[E(D) - g\left(1/\beta_n \right) \right] - \ln\left(n/b_n \right) \to 0 \text{ with } b_n \to b \text{ as } n \to \infty$$

Here $E(D)$ is supposed to be finite and Poisson convergence is generally stated under some specific conditions involving $E(D)$. This leads to the following definition.

Definition 2

Let $E(D)<\infty$ .Then $S_n(\infty)$ is said to obey a Strong Poisson Limit Theorem (SPLT) when it obeys a PLT with a sequence $\{\beta_n\}$ satisfying the condition

$(n+m_n)\beta_n E(D)-\ln\left({}^{n}\!/_{b_n}\right)\to 0$ with $b_n\to b$ as $n\to\infty$.

Consider the Reed-Frost epidemic defined from $\left(n,m_n,\beta_n,E(D)\right)$ and let $\hat{S}_n(\infty)$ denotes the corresponding final number of susceptible. Then from the above propositions $\hat{S}_n(\infty)$ too obeys a PLT.

Corollary 3

Let $E(D)<\infty$. Then a SPLT holds if and only if

$$\ln(x)g(x)\to 0 \text{ as } x\to\infty \qquad \to (19)$$

Proof

Suppose that $S_n(\infty)$ obey a SPLT. Then from

$$(n+m_n)\beta_n\left[E(D)-g\left({}^{1}\!/_{\beta_n}\right)\right]-\ln\left({}^{n}\!/_{b_n}\right)\to 0 \text{ with } b_n\to b \text{ as } n\to\infty$$

$$(n+m_n)\beta_n E(D)-\ln\left({}^{n}\!/_{b_n}\right)\to 0 \text{ with } b_n\to b \text{ as } n\to\infty$$

We get

$$(n+m_n)\beta_n g\left({}^{1}\!/_{\beta_n}\right)\to 0 \qquad \to (20)$$

Again from

$$(n+m_n)\beta_n E(D)-\ln(n/b_n)\to 0 \text{ with } b_n\to b \text{ as } n\to\infty \qquad (20a)$$

We get

$$\ln(n)g\left({}^{1}\!/_{\beta_n}\right)\to 0 \text{ as } n\to\infty \qquad \to (21)$$

Now we may put $m_n=1, n\ge 1$ in (20a). Then we may obtain that $\ln\left({}^{1}\!/_{\beta_n}\right)\sim\ln(n)$ and inserting this in $\ln(n)g\left({}^{1}\!/_{\beta_n}\right)\to 0$ as $n\to\infty$yields $\ln(x)g(x)\to 0$ as $\mathrm{x}\to\infty$.

Conversely, suppose that (19) holds, then clearly (18) holds too, so that $S_n(\infty)$ obey a PLT.

Hence the sequence $\{\beta_n\}$ should satisfy (20) or equivalently (21) which is true by (19).

Hence we have the condition (19) does hold true for the general epidemic and the Reed-Frost epidemic.

Let us consider a stochastic model for the spread of an epidemic among a population of n individuals labelled $1, 2, \ldots n$ in which a typical infected individual i say makes global contacts, with individuals chosen independently and uniformly from the whole population and local contacts with individuals chosen independently according to the contact distribution $v_i^n = \left(v_{i,j}^n; j = 1, 2, \ldots\ldots n\right)$ at the points of independent Poisson process with rates λ_G^n and λ_L^n respectively throughout an infectious period. The population initially comprises m_n infective and $n - m_n$ susceptible. A sufficient condition is derived for the number of individuals who survive the epidemic to converge weakly to a Poisson distribution as $n \to \infty$. This result is specialised to household's model.

From the corollary 3 which is the homogeneously mixing equivalent of the following theorem

Theorem1.

Suppose that there exists $\alpha > 0, 0 < \delta < 1/2$ and $b > 0$ such that $\lambda_L^n n^{-\alpha} \to 0, n^{\delta} g(n) \to \infty$ and $\lambda_G^n E(Q) - \log\left(h_n g(n)\right) + \log b \to 0$ as $n \to \infty$. Suppose in addition that there exists $\varepsilon > \delta + \alpha, 0 < c, d < 1, n_0 \in N$ and for all $1 \le i \le n$, a set of individuals L_i^n in E_n such that for all $n \ge n_0$ and for all $1 \le i \le n, i \in L_i^n$; $\sum_{j \notin L_i^n} v_j^n$, $i < n^{-\varepsilon}$, $\left|L_i^n\right| \le n^c$, $\left|\mu_i^n\right| \le n^d$ and $h_n n^{-(d+2\delta)} \to \infty$ as $n \to \infty$, where $\mu_i^n = \left\{ j; L_i^n \cap L_j^n \ne \varnothing \right\}$,Then $S_n \xrightarrow{D} P_o(b)$ as $n \to \infty$.

Proof

Under the condition of the above theorem $X_n(n) \xrightarrow{D} P_0(b)$ as $n \to \infty$. To prove the above theorem it is sufficient to show that $P\left(x_n(n) \ne S_n\right) \to 0$ as $n \to \infty$. Under the conditions of the above theorem we have $P\left(Y_n(T_n) \ge \left[n^{\delta}\right]\right) \to 0$ as $n \to \infty$. Also we have $P(D_n) \to 0$ as $n \to \infty$. Finally using Jensen's inequality $1 - \varnothing\left(\left[n^{\delta}\right]\beta_G^n\right)^{\left|n^{\delta}\right|} \le 1 - \varnothing\left(\beta_G^n\right)^{n^{2\delta}}$ and since $2\delta < 1$,$1 - \varnothing\left(\beta_G^n\right)^{n^{2\delta}} \to 0$ as $n \to \infty$. Thus $1 - \varnothing\left(\left[n^{\delta}\right]\beta_G^n\right)^{\left|n^{\delta}\right|} \to 0$ as $n \to \infty$. Since for $0 < \delta < 1/2$ such that $n^{\delta} g(n) \to \infty$ as $n \to \infty$. Then for all $n \in N$

$P\left(Y_n\left(T_n\right)\neq X_n\left(n\right)\right)\leq P\left(Y_n\left(T_n\right)>\left[n^{\delta}\right]\right)+P\left(D_n\right)+\left(1-\varnothing\left(\left[n^{\delta}\right]\beta_G^n\right)^{\left|n^{\delta}\right|}\right)$ .The theorem follows.

3. Conclusion

Thus we have developed a Poisson limit theorem for general epidemic. One has to determine R_0^n the mean number of global contacts and that emanate from a typical infectious individual and $g\left(n\right)$ the probability that a randomly chosen initial susceptible has local susceptibility set of size 1.

The theorem implies that the number of survivors of the epidemic is approximately Poisson distributed with mean $h_n g\left(n\right)e^{-R_0^n}$ where n is the total population size and h_n is the initial number of susceptible.

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