Advances in Applied Mathematical Biosciences.
ISSN 2248-9983 Volume 7, Number 1 (2016), pp. 23-31

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Dynamics in a Harvested Prey-Predator Model with Susceptible-Infected-Susceptible (SIS) Epidemic Disease in the Prey

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Abstract

In this paper, a non-linear mathematical model is proposed to study the dynamics of a disease transmission among the prey population. The model includes the harvesting of infected prey. The existence, uniqueness and boundedness of the solution of the model are investigated. The feasibility and the stability conditions of the fixed points of the system are analyzed by applying linearization method and next generation matrix method. Numerical simulations are carried out to justify analytical results.

Keywords: SIS, Epidemic model, Infected prey, next generation matrix, stability analysis.

1. INTRODUCTION

Both the theoretical ecology and the theoretical epidemiology are developed research fields and treated separately. However, there are some common features between these two systems and merging the two areas may show interesting dynamics. Anderson and May were the first who merged the above two fields and formulated a predator-prey model where prey species were infected by some disease [1-2]. In subsequent time, many researchers have proposed and studied different predator-prey models in the presence of disease [3-6].By the end of the 1990s the term eco-epidemiology was first used for the combination of infectious disease and ecological dynamics [11]. Eco-epidemic models describe ecosystems of interacting populations among which a disease spreads [7-9, 14]. It has been shown that invading diseases tend to destabilize the predator-prey communities [2, 10, 15, 17]. A number of mathematical models of disease spread have been introduced relevant to the type of diseases, for example, SI, SIS, SIR [11, 16]. In this paper, we proposed and analyzed a mathematical model describing prey-predator model having SIS epidemic disease in the prey population.

One thing that is important in eco-epidemiological investigations is the reproduction number. It defines the average number of new cases of an infection caused by one typical infected individual in a population consisting of susceptible only. It is one of the means used to determine whether or not an infection can easily be controlled. In this study, the next generation matrix method to determine the reproduction number is proposed.

2. THE MODEL

In this present work on predator prey system, a disease transmission model incorporating provision for harvesting the infected prey has been proposed and studied. We impose the following assumptions to formulate the mathematical model.

(i) It is assumed that a parasite is infectious and it spreads among preys according to a SIS (Susceptible-Infected-Susceptible) model.

(ii) The susceptible prey population grows according to logistic equation.

(iii) The infected prey is harvested.

(iv) The predator population consumes both prey populations.

(v) We have considered Holling type-II functional response for the predation of susceptible prey and since infected preys are easier to catch, Holling type-I is chosen for the predation of infected prey.

(vi) The predator cannot be infected.

(vii) In the absence of prey, predators will experience natural death.

Based on the assumptions, the proposed model is

$$\begin{aligned}
\frac{dx_1}{dt} &= ax_1\left(1-\frac{x_1}{k}\right) - \alpha x_1 x_2 + \beta x_2 - \frac{p_1 x_1 y}{1+sx_1} \\
\frac{dx_2}{dt} &= \alpha x_1 x_2 - \beta x_2 - p_2 x_2 y - hx_2 \qquad (1) \\
\frac{dy}{dt} &= \frac{c_1 p_1 x_1 y}{1+sx_1} + c_2 p_2 x_2 y - dy
\end{aligned}$$

With the initial densities $x_1(0) > 0$, $x_2(0) > 0$ and y(0) > 0. Here x_1(t), x_2(t) and y(t) denote the numbers of Susceptible prey, Infected prey and Predator respectively and parameters are all positive.

Model parameters are described below.

Parameters	Description
a	Logistic growth of susceptible prey (S. Prey)
k	Environmental carrying capacity
α	Rate of contact between susceptible prey and infected prey (I.Prey)
β	Rate of transformation from infected prey to susceptible prey
p_1	Predation rate on susceptible prey
p_2	Predation rate on infected prey
h	Rate of harvesting of infected prey
c_1	Conversion efficiency on susceptible prey
c_2	Conversion efficiency on infected prey
s	Half saturation constant
d	Natural death rate of predator

3. ANALYSIS

It is obvious that the interaction functions in the right hand side of system (1) are continuously differentiable functions in $R_+^3 = \{(x_1, x_2, y) \in R^3, x_1 \geq 0, x_2 \geq 0, y \geq 0\}$ and hence they are Lipschitzian functions. Therefore, the solution of system (1) exists and is unique [18].

3.1 Boundedness of the system

Proposition: 1 All the solutions of the system (1) are uniformly bounded.

Proof: Define a positive definite function W as $W = x_1 + x_2 + y$.

From (1), $\frac{d}{dt}(x_1 + x_2 + y) \leq ax_1(1 - \frac{x_1}{k})$

For arbitrarily chosen η, this simplifies to $\frac{dW}{dt} + \eta W \leq ax_1(1 - \frac{x_1}{k} + \frac{\eta}{a})$.

The above equation has a solution $W \leq \frac{ax_1}{\eta}\left(1 - \frac{x_1}{k} + \frac{\eta}{a}\right)$ $as\ t \longrightarrow \infty$ implying that the solution is bounded for $0 \leq W \leq \frac{ax_1}{\eta}\left(1 - \frac{x_1}{k} + \frac{\eta}{a}\right)$. Therefore, all the solutions of the system (1) are uniformly bounded in the region $\Gamma = \left\{(x_1, x_2, y) \epsilon R_+^3 : W \leq \frac{ax_1}{\eta}\left(1 - \frac{x_1}{k} + \frac{\eta}{a}\right) + \xi\right\} for\ all\ \xi > 0\ and\ t \longrightarrow \infty$.

3.2 Existence of equilibrium points

In this section, the conditions for the existence of all possible equilibrium points of the system (1) are discussed. System (1) results in the following three equilibrium points.

(i) The trivial equilibrium point $E_0\,(0,0,0)$ always exist.

(ii) The axial equilibrium point $E_1(k, 0,0)$ always exist.

(iii) The disease free equilibrium point $E_2(\frac{d}{c_1p_1-ds}, 0, \frac{\alpha d-(\beta+h)(c_1p_1-ds)}{p_2(c_1p_1-ds)})$ exists in the x_1x_2 plane provided $c_1p_1 - ds > 0$ and $\alpha d - (\beta + h)(c_1p_1 - ds) > 0$.

3.3 Stability Analysis

In this section, we analyzed the local stability of the model (1) around each equilibrium point. The condition for the stability of E_0 is derived by applying linearization approach and the conditions for the stability for $E_1 and\ E_2$ are obtained by using the next generation matrix approach introduced in [12].

3.3.1 Linear stability analysis

The Jacobian matrix of the model (1) at state variable is given by

$$J = \begin{bmatrix} a - \frac{2ax_1}{k} - \alpha x_2 - \frac{p_1y}{(1+sx_1)^2} & \alpha x_2 & \frac{c_1p_1y}{(1+sx_1)^2} \\ -\alpha x_1 + \beta & \alpha x_1 - \beta - p_2y - h & c_2p_2y \\ \frac{-p_1x_1}{1+sx_1} & -p_2x_2 & \frac{c_1p_1x_1}{1+sx_1} + c_2p_2x_2 - d \end{bmatrix}$$

The linearized stability technique for analyzing the local behavior of the non-linear system (1) is given in the following theorem.

Theorem

Let $p(\lambda) = \lambda^3 + B\lambda^2 + C\lambda + D$. There are at most three roots of the equation $p(\lambda) = 0$. Then the following statements are true:

a) If every root of the equation has absolute value less than one, then the fixed point of the system is locally asymptotically stable and fixed point is called a sink.
b) If at-least one of the roots of the equation has absolute value greater than one, then the fixed point of the system is unstable and fixed point is called saddle.
c) If every root of the equation has absolute value greater than one, then the system is source.
d) The fixed point of the system is called hyperbolic if no root of the equation has absolute value equal to one. If there exists a root of the equation with absolute value equal to one, then the fixed point is called Non – hyperbolic [13].

- **Stability of Equilibrium E_0**

The Jacobian matrix J(E_0) at the equilibrium point E_0 is given as follows.

$$J(E_0) = \begin{bmatrix} a & 0 & 0 \\ \beta & -\beta - h & 0 \\ 0 & 0 & -d \end{bmatrix}$$

The eigen values are λ_1 = a, λ_2 = -(β+h) and λ_3 = -d. Thus, if a < 1, the equilibrium point E_0 becomes stable. Otherwise, E_0 is unstable. We summarize the result in the following proposition.

Proposition 2: Let $R_{01} = a.$ The equilibrium point E_0 of system (1) is locally asymptotically stable if $R_{01} < 1$ and otherwise, if R_{01} > 1, it is unstable.

3.3.2. The next Generation Matrix method

The non-linear vector function f (x_1, x_2, y) for the system(1) is $f = \mathcal{F} - \mathcal{V}$ where the matrix $\mathcal{F}$ represents the transmission matrix and $\mathcal{V}$ represents the transition matrix. The transmission constitutes all epidemiological events that involve new infections and all other events are incorporated in $\mathcal{V}$.

Hence we have $\mathcal{F} = \begin{bmatrix} 0 \\ \alpha x_1 x_2 \\ 0 \end{bmatrix}$ and $\mathcal{V} = \begin{bmatrix} \frac{a x_1^2}{k} + \left(\frac{p_1 y}{1+s x_1} - a\right) x_1 + (\alpha\, x_1 - \beta) x_2 \\ (\beta + p_2 y + h) x_2 \\ \frac{-p_1 c_1 x_1 y}{1+s x_1} - c_2 p_2 x_2 y + dy \end{bmatrix}$

The Jacobian matrices of the functions $F = D\mathcal{F}$ and $v = D\mathcal{V}$ are obtained as below.

$$F = [f_{ij}]_{3\times 3} = \begin{bmatrix} 0 & \alpha x_2 & 0 \\ 0 & \alpha x_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$V = [v_{ij}]_{3\times 3} = \begin{bmatrix} \frac{2 a x_1}{k} + \frac{p_1 y}{(1+s x_1)^2} - a + \alpha x_2 & \alpha x_1 - \beta & \frac{p_1 x_1}{1+s x_1} \\ 0 & \beta + p_2 y + h & p_2 x_2 \\ -\frac{c_1 p_1 y}{(1+s x_1)^2} & -c_2 p_2 y & -c_2 p_2 x_2 + d \end{bmatrix}$$

We call, FV^{-1}, the next generation matrix for the model and set $\mathcal{R}_0 = \rho(FV^{-1})$, where $\rho(A)$ denotes the spectral radius of a matrix A. By applying the theorem 2 in [12], if x_0 is a disease free equilibrium (DFE) of the model, then x_0 is locally asymptotically stable if $\mathcal{R}_0 < 1$, but unstable if $\mathcal{R}_0 > 1$.

- **Stability of Equilibrium E_1**

The Jacobian matrices F and V for the equilibrium E_1 are as follows.

$$F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha k & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} a & \alpha k - \beta & \frac{p_1 k}{1+ks} \\ 0 & \beta + h & 0 \\ 0 & 0 & d \end{bmatrix}$$

So, $R_{02} = \rho(FV^{-1}) = \frac{f_{22}}{v_{22}} = \frac{\alpha k}{\beta+h}$. Thus, if $R_{02} < 1$, the equilibrium point E_1 becomes stable. Otherwise, E_1 is unstable. We summarize the result in the following proposition.

Proposition 2: Let $R_{02} = \frac{\alpha k}{\beta+h}$. The equilibrium point E_1 of system (1) is locally asymptotically stable if $R_{02} < 1$ and otherwise, if $R_{02} > 1$, it is unstable.

- **Stability of Equilibrium E_2**

The Jacobian matrices F and V for the equilibrium E_2 are as follows.

$$F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\alpha d}{c_1 p_1 - ds} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$V = \begin{bmatrix} \frac{2ad}{(c_1p_1 - ds)k} + \frac{p_1}{p_2}\left[\frac{\alpha a - (\beta + h)(c_1p_1 - ds)}{(c_1p_1 - ds)(1 + sx_1)^2}\right] - a & \frac{\alpha d}{c_1p_1 - ds} - \beta & \frac{d}{c_1} \\ 0 & \frac{\alpha d}{c_1p_1 - ds} & 0 \\ \frac{(ds - c_1p_1)}{c_1p_1p_2}(\alpha d - (\beta + h)(c_1p_1 - ds)) & \frac{c_{2(\alpha d - (\beta+h)(c_1p_1 - ds))}}{ds - c_1p_1} & d \end{bmatrix}$$

So, $R_{03} = \rho(FV^{-1}) = 1$. Thus, if $R_{03} < 1$, the equilibrium point E_2 becomes stable. Otherwise, E_2 is unstable. The following proposition holds.

Proposition 3: Let $R_{03} = \frac{\alpha d}{c_1p_1 - ds}$. The equilibrium point E_2 of system (1) is locally asymptotically stable if $R_{03} < 1$ and otherwise, if $R_{03} > 1$, it is unstable.

Thus, the sufficient conditions for the existence and the local stability of equilibria for system (1) are summarized in the following table.

Equilibria	Existence condition	Stability Condition
$\boldsymbol{E_0}$	Always exists	$R_{01} < 1$
$\boldsymbol{E_1}$	Always exists	$R_{02} < 1$
$\boldsymbol{E_2}$	$c_1p_1 - ds > 0$; $\alpha d - (\beta + h)(c_1p_1 - ds) > 0$	$R_{03} = 1$

4. NUMERICAL SIMULATION

Analytical findings always remain incomplete without numerical verification of the results. In this section, we have performed numerical experiments to study the role of harvesting on the model dynamics. In whole numerical experiment, we use a set of hypothetical parameter values: $a = 1.6$, $\alpha = 0.8$, $h = 0.7$, $c_1 = 0.04$, $c_2 = 0.04$, $\beta = 0.7$, $p_1 = 0.33$, $p_2 = 0.44$, $s = 0.5$, $d = 0.2$ and $k = 100$. It is observed that the increase in harvest affects the disease and thus prevents the occurrence of an epidemic.

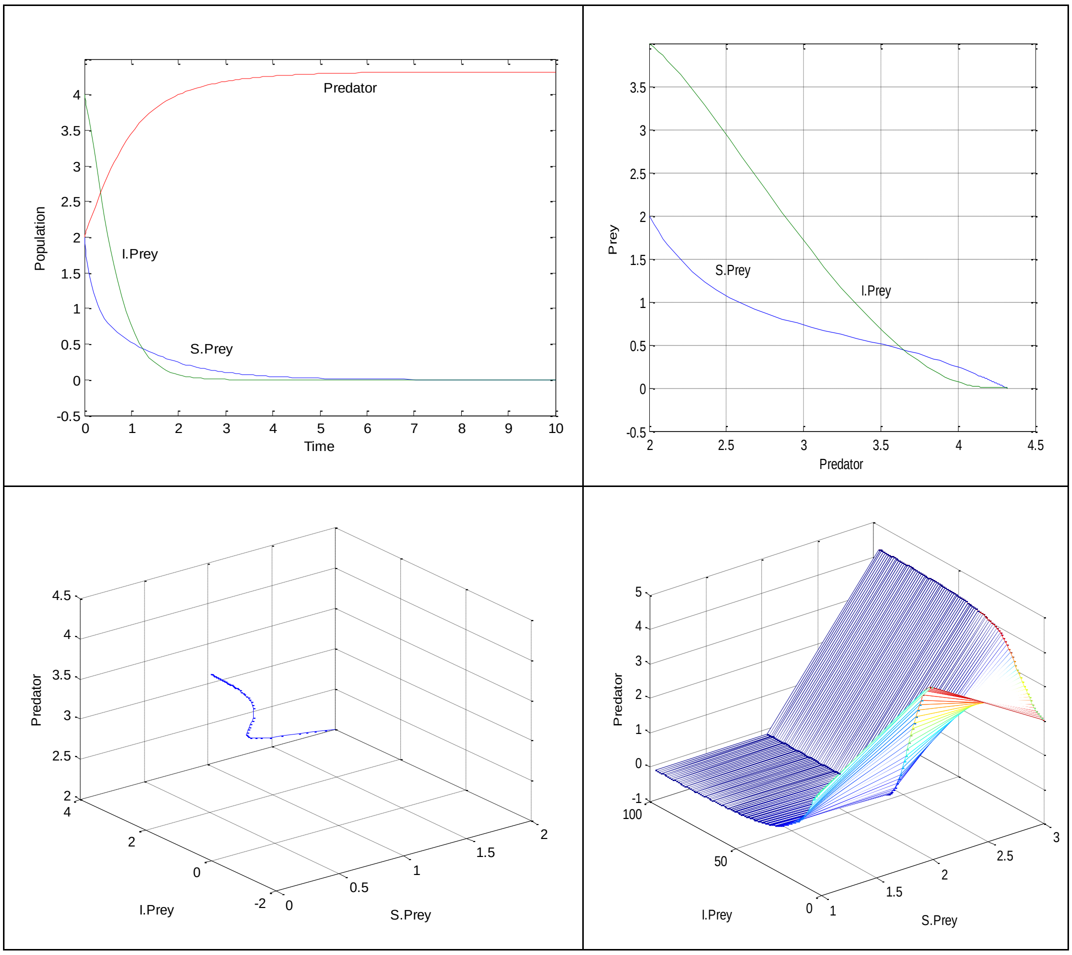

Figure: Proper harvesting strategy helps in making the system disease free

5. CONCLUSION

A harvested prey – predator model with SIS epidemic disease in the prey population is proposed and analyzed. An infectious disease is assumed to spread only among preys with SIS model and predators consume both healthy and infected preys with two different predation response. The boundedness of the trajectories, existence of an attracting set, as well as the existence of equilibria have been analyzed. The dynamic behavior of the system around each equilibrium point has been studied and threshold values for reproduction numbers R_{01}, R_{02} and R_{03} are computed. Numerical Simulations are carried out to confirm the results obtained analytically. The effect of harvesting of infected prey has been analyzed from the point of view of the stability of the system. It is observed that a proper harvesting strategy is needed for the persistence of the species.

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