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Blow-Up Results for the Periodic Peakon b-Family of Equations

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Abstract

The analysis of water waves is an interesting study point in many branches of engineering and science. Partial differential equations particularly play a major role in driving this research and spills over to image processing and pattern recognition. In this paper, a property called "wave breaking" and it's consequences in water waves has been studied analytically. New results on wave breaking of solutions are obtained. Blow-up phenomena for certain profiles have also been discussed.

AMS subject classification:
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1. Introduction

Consider the following partial differential equations.

$$m_t + c_0 m_x + \gamma u_{xxx} + u m_x + b m u_x = 0,$$

with $m = u - \alpha^2 u_{xx}$. The Degasperis-Procesi (DP) equation is a special case of the above

$$y_t + y_x u + 3 y u_x = 0, \quad x \in \mathbb{R}, \ t > 0,$$

with $y = u - u_{xx}$, was originally derived by Degasperis-Procesi [16] using the method of asymptotic integrability up to third order as one of three equations in the family of third order dispersive PDE conservation laws of the form

$$u_t - \alpha^2 u_{xxt} + \gamma u_{xxx} + c_0 u_x = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x. \tag{1.1}$$

The other two integrable equations in the family are the Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + uu_x = 0$$

and the Camassa-Holm (CH) shallow water equation [3,13],

$$y_t + y_x u + 2yu_x = 0, \quad y = u - u_{xx}$$

These three cases exhaust in the completely integrable candidates for (1) by Painlevé analysis. Both the KdV equation [13] and the Camassa-Holm equation [2, 1, 8, 7] are completely integrable models for the propagation of shallow water waves. The DP equation is also in dimensionless space-time variables (x,t) an approximation to the incompressible Euler equations for shallow water under the Kodama transformation [22, 23] and its asymptotic accuracy is the same as that of the Camassa-Holm shallow water equation, where $u(t,x)$ is considered as the fluid velocity at time t in the spatial x-direction with momentum density y. Degasperis, Holm and Hone [15] showed the formal integrability of the DP equation as Hamiltonian systems by constructing a Lax pair and a bi-Hamiltonian structure. The DP equation is observed a model supporting shock waves [29].

The KdV equation is an integrable Hamiltonian equation that possesses smooth solitons as traveling waves. In the KdV equation, the leading order asymptotic balance that confines the traveling wave solitons occurs between nonlinear steepening and linear dispersion. However, the nonlinear dispersion and nonlocal balance in the CH equation and the DP equation, even in the absence of linear dispersion, can still produce a confined solitary traveling waves

$$u(t,x) = ce^{-|x-ct|},$$

traveling at constant speed $c > 0$, which are called the peakons [3, 15]. Peakons of both equations are true solitons that interact via elastic collisions under the CH dynamics, or the DP dynamics, respectively. The peakons of the CH equation are orbitally stable [12].

The DP equation can be rewritten as

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad t > 0,\ x \in \mathbb{R}. \tag{1.2}$$

The peakon solitons are not classical solutions of (2). They satisfy the Degasperis-Procesi equation in the conservation law form

$$u_t + \partial_x \left(\frac{1}{2}u^2 + (1 - \partial_x^2)^{-1} \left(\frac{3}{2}u^2 \right) \right) = 0, \ \ t > 0, \ \ x \in \mathbb{R}. \tag{1.3}$$

More recently, Liu and Yin [27] proved that the first blow-up to Eq. (2) must occur as wave breaking and shock waves possibly appear afterwards. It is shown in [27] that the lifespan of solutions of the DP equation (2) is not affected by the smoothness and size of the initial profiles, but affected by the shape of the initial profiles. This can be viewed as a significant difference between the DP equation (or the CH equation) and the KdV.

It is also noted that the KdV equation, unlike the CH equation or DP equation, does not have wave breaking phenomena, that is, the wave profile remains bounded, but its slope becomes unbounded in finite time [38]. For the CH equation, a procedure to understand the continuation of solutions past wave breaking has been recently presented by Bressan and Constantin in [1]. Yin has also worked on a two component Camassa-Holm equation and proved results on blow-up and existence.

Holm and Staley [22] studied a one-dimensional version of active fluid transport that is described by the following family of $1 + 1$ evolutionary equations

$$m_t + um_x + bu_x m = 0 \qquad with\ u = g * m \tag{1.4}$$

The $um_x + bu_x m$ term in (1.4) is convection stretching and the fluid velocity $u(t, x)$ is defined on the real line vanishing at spatial infinity and $u = g * m$ defines the convolution (filtering)

$$u(x) = \int_{\mathbb{R}} g(x - y)m(y)dy \tag{1.5}$$

which relates velocity u to momentum density m by integration against kernel $g(x)$ over the real line. It was shown by Degasperis & Procesi [16] using the method of asymptotic integrability that equation (1.4) can't be completely integrable unless $b = 2$ or $b = 3$. It is convenient to rewrite (1.4) as

$$u_t - u_{txx} + (b - 1)uu_x = bu_x u_{xx} + uu_{xxx}; \quad t > 0, \ \ x \in \mathbb{S} \tag{1.6}$$

for a real parameter b, which includes both the CS ($b = 2$) and the DP ($b = 3$) equations as special cases. Since it arises from (1.4) when the peakon kernel $g(x) = \frac{1}{2}e^{-|x|}$ is chosen, (1.6) is referred to as the peakon b-family of equations.

The b family equation under a periodic initial profile equation case is the subject of this paper.A number of significant results regarding b family is proved which doesn't take into account only the specific values of $b = 2$ or $b = 3$. The blow-up results established in the paper give precise descriptions of wave-breaking phenomena of the shallow water wave flow in a different direction.

The remainder of the paper is organized as follows. In Section 2, the author proves the local well-posedness of the initial-value problem associated with b family of shallow water wave equations. Section 3 gives the precise blow-up scenario of strong solutions to b family Eq. with certain initial data.

Notation. We denote the norm of the Lebesgue space L^p by $\| \cdot \|_{L^p}$, $1 \le p \le \infty$ and the norm in the Sobolev space H^s, $s \in \mathbb{S}$ by $\| \cdot \|_s$. We denote by $*$ the spatial convolution on $\mathbb{S}$. We also use $(\cdot, \cdot)$ to represent the standard inner product in $L^2(\mathbb{S})$, and $(\cdot, \cdot)_s$, the standard inner product in $H^s(\mathbb{S})$.

2. Local well-posedness

In this section, we apply Kato's theory to establish local well-posedness for the Cauchy problem of (1.6). For convenience, we state here Kato's theorem in a form suitable for our purpose. Consider the abstract quasi-linear evolution equation

$$\begin{cases} \dfrac{dv}{dt} + A(v)v = f(v), \ \ t \geq 0, \\ v(0) = v_0 \end{cases} \tag{2.1}$$

Let X and Y be Hilbert spaces such that Y is continuously and densely embedded in X, and let $Q : Y \to X$ be a topological isomorphism. Let $L(Y, X)$ denote the space of all bounded linear operators from Y to X. If $X = Y$, we denote this space by $L(X)$. The linear operator A belongs to $G(X, 1, \beta)$ where β is a real number, that is, $-A$ generates a C_0-semigroup such that $\|e^{-sA}\|_{L(X)} \leq e^{\beta s}$. We make the following assumptions, where $\mu_1,\ \mu_2,\ \mu_3$, and μ_4 are constants depending only on $\max\{\|y\|_Y, \|z\|_Y\}$.

(i) $A(y) \in L(Y, X)$ for $y \in X$ with

$$\| \left(A(y) - A(z)\right) w\|_X \leq \mu_1 \|y - z\|_X \|w\|_Y, \ \ y, z, w \in Y$$

and $A(y) \in G(X, 1, \beta)$ (i.e., $A(y)$ is quasi-m-accretive), uniformly on bounded sets in Y.

(ii) $QA(y)Q^{-1} = A(y) + B(y)$, where $B(y) \in L(X)$ is bounded, uniformly on bounded sets in Y. Moreover,

$$\| \left(B(y) - B(z)\right) w\|_X \leq \mu_2 \|y - z\|_Y \|w\|_X, \ \ y, z \in Y, w \in X.$$

(iii) $f : Y \to Y$ extends to a map from X into X, is bounded on bounded sets in Y, and satisfies

$$\|f(y) - f(z)\|_Y \leq \mu_3 \|y - z\|_Y, \ \ y, z \in Y,$$

$$\|f(y) - f(z)\|_X \leq \mu_4 \|y - z\|_X, \ \ y, z \in Y.$$

Lemma 2.1. (Kato, [24]) Assume that (i), (ii), and (iii) hold. Given $v_0 \in Y$, there is a maximal $T > 0$ depending only on $\|v_0\|_Y$, and a unique solution v to (2.1) such that $v = v(\cdot, v_0) \in C([0, T), Y) \cap C^1\left([0, T), X\right)$. Moreover, the map $v_0 \mapsto v(\cdot, v_0)$ is a continuous map from Y to $C([0, T), Y) \cap C^1\left([0, T), X\right)$.

We now provide a framework in which we shall reformulate the problem (1.2). We begin by fixing some notations. All spaces of functions are assumed to be over $\mathbb{S}$, where $S = \mathbb{R} \setminus Z$ for simplicity, we drop $\mathbb{S}$ in our notation for function spaces if there is no ambiguity. If A is an unbounded operator, we denote by $D(A)$ the domain of A. $[A, B]$ denotes the commutator of two linear operators A and B.

With $m = u - u_{xx}$, we consider the Cauchy problem

$$\begin{cases} m_t + um_x + bu_x m = 0, & t > 0,\ x \in \mathbb{S}, \\ m(0, x) = u_0(x) - u_{0,xx}(x), & x \in \mathbb{S}, \\ u(t, x) = u(t, x+1). \end{cases} \tag{2.2}$$

Note that if $g(x) := \frac{1}{2} e^{-|x|},\ x \in \mathbb{S}$, then $(1 - \partial_x^2)^{-1} f = g * f$ for all $f \in L^2(\mathbb{S})$ and $g * m = u$, where $*$ denotes convolution. Using this identity, we can rewrite (2.2) as a quasi-linear evolution equation of hyperbolic type:

$$\begin{cases} u_t + uu_x + \partial_x g * \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 \right) = 0, & t > 0,\ x \in \mathbb{S}, \\ u(0, x) = u_0(x), & x \in \mathbb{S}, \\ u(t, x) = u(t, x+1). \end{cases} \tag{2.3}$$

Definition 2.2. If $u \in C([0, T), H^s(\mathbb{S})) \cap C^1([0, T), H^{s-1}(\mathbb{S}))$ with $s > \frac{3}{2}$ satisfies (2.3), then u is called a strong solution to (2.3). If u is a strong solution on $[0, T)$ for every $T > 0$, then it is called global strong solution to (2.3).

The local well-posedness of the Cauchy problem of (2.3) with initial data $u_0 \in H^s(\mathbb{S}),\ s > \frac{3}{2}$ can be obtained by applying Kato's theorem [24]. More precisely, we have the following well-posedness result.

Theorem 2.3. For any constant b, given $u_0 \in H^s(\mathbb{S}),\ s > \frac{3}{2}$, there exist a maximal $T = T(u_0) > 0$ and a unique strong solution u to (2.3), such that

$$u = u(\cdot, u_0) \in C([0, T), H^s(\mathbb{S})) \cap C^1([0, T), H^{s-1}(\mathbb{S})).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $u_0 \mapsto u(\cdot, u_0) : H^s(\mathbb{S}) \to C([0, T), H^s(\mathbb{S})) \cap C^1([0, T), H^{s-1}(\mathbb{S}))$ is continuous.

To prove this theorem, we apply Lemma 2.1 with $A(u) = u\partial_x,\ f(u) = -\partial_x g * \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 \right),\ Y = H^s,\ X = H^{s-1},\ \Lambda = (1 - \partial_x^2)^{\frac{1}{2}}$, and $Q = \Lambda^1$. Obviously, Q is an isomorphism of H^s onto H^{s-1}. Thus, in order to derive Theorem 2.3 from Lemma 2.1, we only need to verify that $A(u)$ and $f(u)$ satisfy the conditions (i)-(iii). We break the argument into several lemmas.

Lemma 2.4. **([36])** The operator $A(u) = u\partial_x$, with $u \in H^s, s > \frac{3}{2}$, belongs to $G(H^{s-1}, 1, \beta)$.

Lemma 2.5. Let the operator $A(u) = u\partial_x$ with $u \in H^s, s > \frac{3}{2}$. Then $A(u) \in L(H^s,\ H^{s-1})$ for $u \in H^s$. Moreover,

$$\| \left(A(u) - A(z)\right) w\|_{s-1} \le \mu_1 \|u - z\|_{s-1} \|w\|_s, \quad u, z, w \in H^s. \tag{2.4}$$

Proof. Let $u, z, w \in H^s, s > \frac{3}{2}$. Note that H^{s-1} is a Banach algebra. Then we have

$$\| \left(A(u) - A(z)\right) w\|_{s-1} \le c\|u - z\|_{s-1} \|\partial_x w\|_{s-1} \le \mu_1 \|u - z\|_{s-1} \|w\|_s.$$

Taking $z = 0$ in the above inequality, we obtain $A(u) \in L(H^s, H^{s-1})$. This completes the proof of Lemma 2.5. ∎

Lemma 2.6. ([37]) $B(u) = [\Lambda^1, u\partial_x]\Lambda^{-1} \in L(H^{s-1})$ for $u \in H^s,\ s > 3/2$. Moreover,

$$\| \left(B(u) - B(z)\right) w\|_{s-1} \le \mu_2 \|u - z\|_s \|w\|_{s-1}, \quad u, z \in H^s, w \in H^{s-1}.$$

Lemma 2.7. Let $f(u) = -\partial_x g * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right)$. Then, f is bounded on bounded set in H^s and satisfies

$$\begin{aligned} &(a)\ \|f(y) - f(z)\|_s \le \mu_3 \|y - z\|_s, \quad y, z \in H^s,\\ &(b)\ \|f(y) - f(z)\|_{s-1} \le \mu_4 \|y - z\|_{s-1}, \quad y, z \in H^s. \end{aligned} \tag{2.5}$$

Proof. Let $y, z \in H^s,\ s > \frac{3}{2}$. Since H^{s-1} is a Banach algebra, it follows that

$$\begin{aligned} &\|f(y) - f(z)\|_s\\ &= \left\| -\partial_x(1 - \partial_x^2)^{-1} \left(\frac{b}{2}(y^2 - z^2) + \frac{3-b}{2}(y_x^2 - z_x^2)\right) \right\|_s\\ &\le \frac{|b|}{2} \|(y - z)(y + z)\|_{s-1} + \frac{|3-b|}{2} \|(y_x - z_x)(y_x + z_x)\|_{s-1}\\ &\le \frac{|b|}{2} \left(\|y - z\|_{s-1} \|y + z\|_{s-1}\right) + \frac{|3-b|}{2} \left(\|y_x - z_x\|_{s-1} \|y_x + z_x\|_{s-1}\right)\\ &\le c\|y - z\|_s \left(\|y\|_s + \|z\|_s\right) \end{aligned}$$

This proves (a). Taking $z = 0$ in the above inequality, we obtain that f is bounded on any bounded set in H^s.

On the other hand, let $y, z \in H^s,\ s > \frac{3}{2}$. Then we have

$$\begin{aligned}
&\|f(y) - f(z)\|_{s-1} \\
&= \left\| -\partial_x (1-\partial_x^2)^{-1} \left(\frac{b}{2}(y^2 - z^2) + \frac{3-b}{2}(y_x^2 - z_x^2) \right) \right\|_{s-1} \\
&\leq \frac{|b|}{2} \|(y-z)(y+z)\|_{s-1} + \frac{|3-b|}{2} \|(y_x - z_x)(y_x + z_x)\|_{s-2} \\
&\leq \frac{|b|}{2} \left(\|y-z\|_{s-1} \|y+z\|_{s-1} \right) + \frac{|3-b|}{2} \left(\|y_x - z_x\|_{s-2} \|y_x + z_x\|_{L^\infty} \right) \\
&\leq \frac{|b|}{2} \left(\|y-z\|_{s-1} \|y+z\|_{s-1} \right) + \frac{|3-b|}{2} \left(\|y_x - z_x\|_{s-2} \|y_x + z_x\|_{s-1} \right) \\
&\leq c\|y-z\|_{s-1} \left(\|y\|_s + \|z\|_s \right).
\end{aligned}$$

where use has been made of the Sobolev imbedding theorem $H^{s-1} \hookrightarrow L^\infty$. This completes the proof of Lemma 2.7. ■

Proof [Proof of Theorem 2.3.]. Theorem 2.3 follows from Lemma 2.1 and Lemmas 2.4-2.7. ■

Theorem 2.8. The maximal T in Theorem 2.3 may be chosen independent of s in the following sense. If $u = u(\cdot, u_0) \in C\left([0,T), H^s\right) \cap C^1\left([0,T), H^{s-1}\right)$ to (2.3) and $u_0 \in H^{s'}$ for some $s' \neq s$, $s' > \frac{3}{2}$, then $u \in C\left([0,T), H^{s'}\right) \cap C^1\left([0,T), H^{s'-1}\right)$ and with the same T. In particular, if $u_0 \in H^\infty = \bigcap_{s \geq 0} H^s$, then $u \in C\left([0,T), H^\infty\right)$.

To prove this theorem, we need the following lemma.

Lemma 2.9. ([24]) Let s, t be real number such that $-s < t \leq s$. Then $L_s^p \bigcap L^\infty$ is an algebra. Moreover,

$$\|fg\|_t \leq c\|f\|_s |g|_t, \quad if \;\; s > \frac{1}{2}$$
$$\|fg\|_{s+t-\frac{1}{2}} \leq c\|f\|_s |g|_t, \quad if \;\; s < \frac{1}{2}$$

where c is a positive constant depending on s and t.

Lemma 2.10. ([25]) Let $f \in H^r, r > \frac{3}{2}$ and let M_f be the multiplication operator by f. Then $\Lambda^{-\tilde{s}}[\Lambda^{\tilde{s}+\tilde{t}+1}, M_f]\Lambda^{-\tilde{t}} \in L\left(L^2(\mathbb{R}^2)\right)$, if $|\tilde{s}|, |\tilde{t}| \leq r-1$. Moreover,

$$\|\Lambda^{-\tilde{s}}[\Lambda^{\tilde{s}+\tilde{t}+1}, M_f]\Lambda^{-\tilde{t}}\|_{L(L^2)} \leq c\|\partial f\|_{r-1}.$$

Proof [Proof of Theorem 2.8.]. It suffices to consider the case $s' > s$, since the case $s' < s$ is obvious from the uniqueness of solutions which is guaranteed by Theorem 2.3. Also we can suppose that $s < s' \leq s + 1$. Since if $s' > s + 1$, we obtain the result by iterated application of the below argument.

If we apply operator Λ^2 to (2.3), we obtain the following evolution equation for $m(t) = \Lambda^2 u(t) = u - u_{xx}$:

$$\begin{cases} \dfrac{d}{dt}m(t) + A(t)m + B(t)m = 0 \\ m(0) = \Lambda^2 u(0) \end{cases} \tag{2.6}$$

where $A(t)m = \partial_x(um)$, $B(t)m = (b-1)u_x m$ and $u \in C\left([0,T), H^s\right)$ is viewed as a known function. Note also that $m \in C\left([0,T), H^{s-2}\right)$ and $m(0) = \Lambda^2 u(0) \in H^{s'-2}$. Our objective is to prove that $m \in C\left([0,T), H^{s'-2}\right)$ which will imply $u \in C\left([0,T), H^{s'}\right)$, because $(1 - \partial_x^2)$ is an isomorphism from $H^{s'}$ to $H^{s'-2}$. This will complete the proof of Theorem 2.8.

Note that $u \in C\left([0,T), H^s\right), u_x \in H^{s-1}$, and that H^{s-1} is a Banach algebra. Then we obtain $B(t) \in L(H^{s-1})$.

To accomplish this, following the argument in Lemmas 3.1-3.3 in [24]) we first need to prove that the family $A(t)$ has a unique evolution operator $\{U(t,\tau)\}$ associated with the spaces $X = H^h$ and $Y = H^k$, where $-s \leq h \leq s-2, 1-s \leq k \leq s-1$, and $k \geq h+1$. Therefore, according to the proof of Lemma 3.1 in [24], we need to verify the following three conditions.

(i) $A(t) \in G(H^h, 1, \beta)$.

(ii) $\Lambda^h \partial_x [\Lambda^{k-h}, u] \Lambda^{-k}$ is uniformly bounded on L^2.

(iii) $A(t) \in L(H^k, H^h)$ is strongly continuous in t.

Let us begin verifying (i). Due to H^h being a Hilbert space, $A(t) \in G(H^h, 1, \beta)$ that is, we will show the following conditions (cf. [24])

(a) $(A(t)w, w)_h \geq -\beta \|w\|_h^2$,

(b) $-A(t)$ is the infinitesimal generator of a C_0-semigroup on H^h, for some (or all)$\lambda > \beta$.

To prove (a), note that

$$\left(\Lambda^h \partial_x(u(t)w), \Lambda^h w\right)_0 \geq -\beta \|w\|_h^2. \tag{2.7}$$

We begin estimating the term on the left-hand side of this inequality, which can be written of the following way

$$-\left(\Lambda^h(uw), \partial_x \Lambda^h w\right)_0, \tag{2.8}$$

but we have

$$\begin{aligned}\Lambda^h(uw) &= \Lambda^h\left(u\Lambda^{-h}(\Lambda^h w)\right) = \Lambda^h\left(\Lambda^{-h}(u\Lambda^h w) - [\Lambda^{-h}, u]\Lambda^h w\right)\\ &= u\Lambda^h w - \Lambda^h[\Lambda^{-h}, u]\Lambda^h w.\end{aligned} \tag{2.9}$$

Using the identity in (2.8), we obtain

$$\begin{aligned}&-\left(\Lambda^h(uw), \partial_x\Lambda^h w\right)_0\\ &= \left(\Lambda^h[\Lambda^{-h}, u]\Lambda^h w, \partial_x\Lambda^h w\right)_0 - \left(u\Lambda^h w, \partial_x\Lambda^h w\right)_0\\ &= \left(\Lambda^{h+1}[\Lambda^{-h}, u]\Lambda^h w, \partial_x\Lambda^{h-1} w\right)_0 - \left(u\Lambda^h w, \partial_x\Lambda^h w\right)_0.\end{aligned} \tag{2.10}$$

The second term on the right-hand side of (2.10) can be easily estimated by applying the Cauchy-Schwartz inequality and integration by parts. For the other we apply the Cauchy-Schwartz inequality and Lemma 2.10 (with $r = s > \frac{3}{2}, \tilde{s} = -h-1, \tilde{t} = 0$). Thus (2.7) is proved.

Next we verify (b). Let $S = \Lambda^{s-1-h}$. Note that S is an isomorphism of H^{s-1} onto H^h and that H^{s-1} is continuously and densely embedded in H^h as $-s \le h \le s-2$. Define

$$\begin{aligned}A_1(t) &:= SA(t)S^{-1} = \Lambda^{s-1-h}A(t)\Lambda^{1+h-s},\\ B_1(t) &:= A_1(t) - A(t) = [S, A(t)]S^{-1}.\end{aligned}$$

Let $w \in H^h$ and $u \in H^s, s > \frac{3}{2}$. Then it follows from Lemma 2.10 with $\tilde{s} = -(h+1), \tilde{t} = s-1$ that

$$\begin{aligned}\|B_1(t)w\|_h &= \left\|\Lambda^h\partial_x[\Lambda^{s-1-h}, u]\Lambda^{1+h-s}w\right\|_0\\ &\le \left\|\Lambda^h\partial_x[\Lambda^{s-1-h}, u]\Lambda^{1-s}\right\|_{L(L^2)}\|\Lambda^h w\|_0\\ &\le \|u\|_s\|w\|_h.\end{aligned}$$

Therefore, we have $B_1(t) \in L\left(H^h\right)$. Since $A(t)m = \partial_x(um) = u\partial_x m + \partial_x u m$, and $\partial_x u \in L\left(H^{s-1}\right)$, by applying Lemma 2.4 and a perturbation theorem for semigroups, we see that H^{s-1} is $A(t)-$admissible. Further, applying Lemma 5.3 in [36] (also see [33]) with $Y = H^{s-1}, X = H^h$ and $S = \Lambda^{s-1-h}$, we obtain that $-A_1(t)$ is the infinitesimal generator of a C_0-semigroup on H^h. Since $A_1(t) = A(t) + B_1(t)$ and $B_1(t) \in L\left(H^h\right)$, by a perturbation theorem for semigroups it follows that $-A(t)$ is the infinitesimal generator of a C_0-semigroup on H^h. This proves (b).

Next, we show (ii). But this is again a consequence from Lemma 2.10, since H^{k-h} is the isomorphism of Y onto X and we have the following estimate

$$\left\|\Lambda^h\partial_x[\Lambda^{k-h}, u]\Lambda^{-k}w\right\|_0 \le c\|u\|_s\|w\|_0.$$

Finally, we verify (iii). In fact, consider the continuity of u and the following inequality

$$\begin{aligned} \| (A(t+\Delta t)-A(t))\, w\|_h &= \|\partial_x \left((u(t+\Delta t)-u(t))w\right)\|_h \\ &\le \| (u(t+\Delta t)-u(t))\, w\|_{h+1} \\ &\le \|u(t+\Delta t)-u(t)\|_{s-1}\|w\|_{h+1} \\ &\le \|u(t+\Delta t)-u(t)\|_{s}\|w\|_{k}. \end{aligned} \tag{2.11}$$

In view of the second inequality in Lemma 2.9, it is easy to see (iii) holds. Thus, the above three conditions imply the existence and uniqueness of evolution operator $U(t,\tau)$ for the family $A(t)$. In particular $U(t,\tau)$ maps H^r into itself for $-s \le r \le s-1$.

Choose $Y = H^{s-2}, X = H^{s-3}$ and note that $m \in C\left([0,T), H^{s-1}\right) \cap C^1\left([0,T), H^{s-2}\right)$. By the properties of evolution operator $U(t,\tau)$, we deduce that

$$\frac{d}{d\tau}\left(U(t,\tau)m(\tau)\right) = -U(t,\tau)B(\tau)m(\tau) \tag{2.12}$$

An integration in $\tau \in [0,t]$ yields

$$m(t) = U(t,0)m(0) - \int_0^t U(t,\tau)B(\tau)m(\tau)d\tau. \tag{2.13}$$

If $s < s' \le s+1$, then we have that $B(t) = \partial_x u \in L(H^{s'-2})$ is strongly continuous in $[0,t)$, and that $H^{s-1}\ H^{s'-2} \subset H^{s'-2}$ by $s-1 > \frac{1}{2}$ (this is a consequence of Lemma 2.9). Due to $-s < s-2 < s'-2 \le s-1$, the family $\{U(t,\tau)\}$ is strongly continuous on $H^{s'-2}$ to itself. Observe that $m(0) \in H^{s'-2}$, we have only see (2.13) as an integral equation of Volterra type, which can be solved for m by successive approximation. This completes the proof of Theorem 2.8. ■

3. Blow-up

By using the local well-posedness result of Theorem 2.3 and energy estimates, the following precise blow-up scenario of strong solutions to (2.3) can be obtained.

Theorem 3.1. Assume $b \in \mathbb{R}$ and $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$. Then blow up of the strong solution $u = u(\cdot, u_0)$ in finite time $T < +\infty$ occurs if and only if

$$\liminf_{t\uparrow T}\{(2b-1)\inf_{x\in\mathbb{R}}[u_x(t,x)]\} = -\infty.$$

Case 1: If b=$\frac{1}{2}$.

Proof.

$$\frac{d}{dt}\int_{\mathbb{S}} m^2 dx = 0 \implies \frac{d}{dt}\int_{\mathbb{S}} m^2 dx = \frac{d}{dt}\int_{\mathbb{S}} m_0^2 dx \quad \forall t \in [0,T) \tag{3.1}$$

On the other hand

$$|u_x| \leq \frac{1}{\sqrt{2}}\|u\|_{H^2} \leq \frac{1}{\sqrt{2}}\|m\|_{L^2} = \frac{1}{\sqrt{2}}\|m_0\|_{L^2}$$

Then, by 3.2

$$\frac{d}{dt}\int_{\mathbb{S}} m_x^2 dx = -2\int_{\mathbb{S}} m_x^2 u_x dx \ + \frac{1}{2}\int_{\mathbb{S}} m^2 u_x dx \leq \sqrt{2}\|m_0\|_{L^2}\int_{\mathbb{S}} m_x^2 dx \ + \frac{1}{2\sqrt{2}}\|m_0\|_{L^2}^3$$

By means of Gronwall's inequality,

$$\|m_x\|_{L^2}^2 \leq (\|\partial_x m_0\|_{L^2}^2 + \frac{1}{4}\|m_0\|_{L^2}^2)\, e^{\sqrt{2}\|m_0\|_{L^2} t}$$

■

Case 2: if b $\neq \frac{1}{2}$ and if u_x is bounded.

Proof. Applying a simple density argument, we only need to show that the above theorem with some $s \geq 3$. Here we assume $s = 3$ to prove the above theorem. Multiplying (2.2) with m and integrating on $\mathbb{S}$ with respect to x, we obtain

$$\frac{d}{dt}\int_{\mathbb{S}} m^2 dx = -(2b-1)\int_{\mathbb{S}} m^2 u_x dx \tag{3.2}$$

On the other hand, differentiating (2.2) with respect to x and multiplying with m_x, integrating on $\mathbb{S}$ with respect to x, and integrating by parts yield

$$\frac{d}{dt}\int_{\mathbb{S}} m_x^2 dx = -(2b+1)\int_{\mathbb{S}} m_x^2 u_x dx + b\int_{\mathbb{S}} m^2 u_x dx. \tag{3.3}$$

It is thereby inferred from (3.2) and (3.3) that

$$\frac{d}{dt}\int_{\mathbb{S}} (m^2 + m_x^2) dx = -(2b+1)\int_{\mathbb{S}} m_x^2 u_x dx - (b-1)\int_{\mathbb{S}} m^2 u_x dx. \tag{3.4}$$

If u_x is bounded from below on $[0,T) \times \mathbb{S}$, i.e., there exists $M > 0$ such that

$$-u_x(t,x) \leq M \quad \textit{on } [0,T) \times \mathbb{S},$$

then the relation (3.3) implies

$$\frac{d}{dt}\int_{\mathbb{S}}\left(m^2+m_x^2\right)dx \le (2b+1)M\int_{\mathbb{S}}\left(m^2+m_x^2\right)dx.$$

And by means of Gronwall's inequality, we deduce that

$$\int_{\mathbb{S}} m^2+m_x^2dx \le \left(\int_{\mathbb{S}} m_0^2+m_{0x}^2dx\right)e^{(2b+1)Mt},\ \forall t\in[0,T). \tag{3.5}$$

Noting that

$$\|u\,(t)\|_3 \le \left(\int_{\mathbb{S}} m^2+m_x^2dx\right)^{1/2}$$

and in view of (3.5), it follows that if $\{u_x\,(t)\}$ is bounded from below on $[0,T)$, then the $H^3(\mathbb{S})$-norm of the solution to Eq.(2.3) is said not to have broken in finite time. This completes the proof of Theorem 3.1. ■

Let us now consider the following differential equation

$$\begin{cases} q_t = u(t,q), & t\in[0,T),\\ q(0,x)=x, & x\in\mathbb{S}. \end{cases} \tag{3.5}$$

Applying classical results in the theory of ordinary differential equations, we have the following properties of q which are crucial in the proof of global existence. The consideration of (3.5) is geometrically motivated for the Camassa-Holm equation [3, 13]. Such a geometric interpretation is lacking for the peakon b-family of equations with $b\ne 2$, but nevertheless some important invariance properties can be deduced [21, 32].

Lemma 3.2. Let $u_0\in H^s(\mathbb{S}),\ s>\frac{3}{2}$, and let $T>0$ be the maximal existence time of the corresponding strong solution u to Eq.(2.3). Then the Eq. (3.5) has a unique solution $q\in C^1([0,T)\times\mathbb{S},\mathbb{S})$ such that the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{S}$ with

$$q_x(t,x)=\exp\left(\int_0^t u_x(s,q(s,x))ds\right)>0,\ \ \forall(t,x)\in[0,T)\times\mathbb{S}. \tag{3.6}$$

Furthermore, setting $m=u-u_{xx}$, we have

$$m(t,q(t,x))q_x^b(t,x)=m_0(x),\quad \forall(t,x)\in[0,T)\times\mathbb{S}. \tag{3.7}$$

Proof. Since $u\in C^1\left([0,T),H^{s-1}(\mathbb{S})\right)$ and $H^s(\mathbb{S})\hookrightarrow C^1(\mathbb{S})$, we see that both functions $u(t,x)$ and $u_x(t,x)$ are bounded, Lipschitz in the space variable x, and of class C^1 in time. Therefore, for fixed $x\in\mathbb{S}$, Eq. (3.5) is an ordinary differential equation. Then well-known classical results in the theory of ordinary differential equation yield that Eq. (3.5) has a unique solution $q(t,x) \in C^1\left([0,T) \times \mathbb{S}; \mathbb{S}\right)$.

Differentiation of Eq.(3.5) with respect to x yields

$$\begin{cases} \dfrac{d}{dt} q_x = u_x(t,q) q_x, \quad t \in [0,T), \\ q_x(0,x) = 1, x \in \mathbb{S}. \end{cases} \tag{3.8}$$

The solution to Eq.(3.8) is given by

$$q_x(t,x) = \exp\left(\int_0^t u_x(s,q(s,x))ds\right), \ (t,x) \in [0,T) \times \mathbb{S}. \tag{3.9}$$

For every $T' < T$, it follows from the Sobolev imbedding theorem that

$$\sup_{(s,x)\in[0,T')\times\mathbb{S}} |u_x(s,x)| < \infty.$$

We infer from (3.9) that there exists a constant $K > 0$ such that $q_x(t,x) \geq e^{-Kt}$, $(t,x) \in [0,T) \times \mathbb{R}$, which implies that the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with

$$q_x(t,x) = \exp\left(\int_0^t u_x(s,q(s,x))ds\right) > 0, \ \forall (t,x) \in [0,T) \times \mathbb{S}.$$

On the other hand, combining (3.8) with (2.2), we have

$$\begin{aligned} \frac{d}{dt}\left(m(t,q(t,x))q_x^b(t,x)\right) &= (m_t + m_x q_x)\, q_x^b(t,x) + bmq_x^{b-1} q_{xt} \\ &= q_x^b(m_t + m_x u + bu_x m) = 0. \end{aligned} \tag{3.10}$$

So,

$$m(t,q(t,x))q_x^b(t,x) = m_0(x), \quad \forall (t,x) \in [0,T) \times \mathbb{S}. \tag{3.11}$$

This completes the proof of Lemma 3.2. ■

Remark 3.3. Lemma 3.2 shows that, if $m_0 = u_0 - u_{0xx}$ does not change sign, then $m\,(t)$ $(\forall\, t)$ will not change sign, as long as $m\,(t)$ exists.

Theorem 3.4. (***The blow-up theorem***) Let $1 < b \leq 3$ and $u_0 \in H^s(\mathbb{S})$, $s > \dfrac{3}{2}$ be odd and nonzero. If $u_{0x}(0) \leq 0$, then the corresponding solution blows up in finite time.

Proof. Again, applying a simple density argument, we only need to show that the above theorem with some $s \geq 3$. Since u_0 is odd, then $u(t,x)$ is odd and $u(t,0) = u_{xx}(t,0) = 0$. Taking derivatives with respect to x on both sides of (2.3), we obtain

$$\begin{aligned} u_{xt} &= -u_x^2 - uu_{xx} - \partial_x^2 g * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right) \\ &= \frac{b}{2}u^2 - \frac{b-1}{2}u_x^2 - uu_{xx} - g * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right) \end{aligned} \tag{3.12}$$

Taking values of (3.12) at $x = 0$ and letting $h(t) = u_x(t, 0)$, it is deduced that

$$\frac{dh}{dt} = -\frac{b-1}{2}h^2 - g * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right).$$

If $h(0) = u_{0x}(0) < 0$, then

$$\frac{dh}{dt} \leq -\frac{b-1}{2}h^2.$$

So,

$$h(t) \leq \frac{1}{\frac{b-1}{2}t + \frac{1}{h(0)}},$$

which tends to $-\infty$ as t goes to $-\dfrac{1}{h(0)}$.

If $h(0) = u_{0x}(0) = 0$, by the continuity of the ordinary differential equation and the uniqueness, we have

$$\frac{dh}{dt} \leq -g * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right),$$

and consequently $h(t) < 0$, for all $t > 0$. So we can choose some time $t_0 > 0$, such that

$$\frac{dh}{dt} \leq -\frac{b-1}{2}h^2$$

for $t > t_0$, and $h(t_0) < 0$. Then we get the finite time blow-up result by the previous discussion. ■

In contrast with the conditions of the blow-up solution of the DP equation defined on the line $\mathbb{R}$, one can see that the criteria of blow-up for periodic solutions of the DP equation are quite different. Let us consider periodic solutions, i.e, $u : \mathbb{S} \times [0, T) \to \mathbb{R}$ where $\mathbb{S}$ is the unit circle and $T > 0$ is the maximal existence time of the solution. The interest in periodic solutions is motivated by the observation that the majority of the waves propagating on a channel are approximately periodic.

Define $G(x)$ by $G(x) = \dfrac{\cosh(x - [x] - \frac{1}{2})}{2\sinh(\frac{1}{2})}$, where $[x]$ stands for the integer part of $x \in \mathbb{R}$, then $(1 - \partial_x^2)^{-1} f = G * f$ for all $f \in L^2(\mathbb{S})$. Using this identity, we can rewrite b-family eq. as a quasi-linear evolution equation of hyperbolic type, namely,

$$\begin{cases} u_t + uu_x + \partial_x G * \left(\frac{3}{2}u^2 + \frac{3-b}{2}u_x^2\right) = 0, & t > 0,\ x \in \mathbb{S}, \\ u(0, x) = u_0(x), & x \in \mathbb{S}, \\ u(t, x) = u(t, x+1), & t \geq 0,\ x \in \mathbb{S}, \end{cases} \tag{3.13}$$

Theorem 3.5. Let $\frac{5}{3} < \mathrm{b} \leq 3$ and $\int_{\mathbb{S}} u_x^3(0)\,dx < 0$. Assume that $u_0 \in H^s(\mathbb{S}), s > \frac{3}{2}$, $u_0 \not\equiv 0$, and the corresponding solution $u(t)$ of (3.13) has a zero for any time $t \geq 0$. Then, the solution $u(t)$ of the above equation (3.13) blows up in finite time.

Proof. Proof of blow-up solution for the periodic case is quite different from that of the line case. By assumption, for each $t \in [0, T)$ there is a $\xi_t \in [0, 1]$ such that $u(t, \xi_t) = 0$. Then for $\forall x \in \mathbb{S}$ we have

$$u^2(t,x) = \left(\int_{\xi_t}^{x} u_x\,dx\right)^2 \leq (x - \xi_t) \int_{\xi_t}^{x} u_x^2\,dx, \quad x \in \left[\xi_t, \xi_t + \frac{1}{2}\right]. \tag{3.14}$$

Hence the above relation and an integration by parts yield

$$\int_{\xi_t}^{\xi_t+\frac{1}{2}} u^2 u_x^2\,dx \leq \int_{\xi_t}^{\xi_t+\frac{1}{2}} (x-\xi_t) u_x^2 \left(\int_{\xi_t}^{x} u_x^2\right) dx \leq \frac{1}{4}\left(\int_{\xi_t}^{\xi_t+\frac{1}{2}} u_x^2\,dx\right)^2.$$

Combining this estimate with a similar estimate on $[\xi_t + \frac{1}{2}, \xi_t + 1]$, we obtain

$$\int_{\mathbb{S}} u^2 u_x^2\,dx \leq \frac{1}{4}\left(\int_{\mathbb{S}} u_x^2\,dx\right)^2. \tag{3.15}$$

We also have

$$\sup_{x\in\mathbb{S}} u^2(t,x) \leq \frac{1}{2}\int_{\mathbb{S}} u_x^2\,dx. \tag{3.16}$$

Let us assume that the solution $u(t,x)$ exists globally in time. Note that $G(x) \geq \frac{1}{2\sinh(\frac{1}{2})}$ for all $x \in \mathbb{S}$. Then

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx &= 3\int_{\mathbb{S}} u_x^2\left(-\frac{b-1}{2}u_x^2 - uu_{xx} + \frac{b}{2}u^2 - G*\left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right)\right) dx \\
&= -3\frac{b-1}{2}\int_{\mathbb{S}} u_x^4\,dx - 3\int_{\mathbb{S}} u_x^2 uu_{xx}\,dx + \frac{3b}{2}\int_{\mathbb{S}} u_x^2 u^2\,dx - 3\int_{\mathbb{S}} u_x^2 G*\left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right) dx \\
&= -3\frac{b-1}{2}\int_{\mathbb{S}} u_x^4\,dx + \int_{\mathbb{S}} u_x^4\,dx + \frac{3b}{8}\left(\int_{\mathbb{S}} u_x^2\,dx\right)^2 - 3\int_{\mathbb{S}} u_x^2 G*\left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right) dx
\end{aligned} \tag{3.17}$$

Since $G \geq \frac{1}{2sh(\frac{1}{2})} > 0$, and $0 \leq b \leq 3$, we have

$$\begin{aligned} &-\int_{\mathbb{S}} u_x^2 G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right) dx \\ &\leq -\frac{3}{2sh(\frac{1}{2})}\int_{\mathbb{S}} u_x^2\, dx \int_{\mathbb{S}} \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right) dx \\ &\leq -\frac{3}{4}\frac{3-b}{sh(\frac{1}{2})}\int_{\mathbb{S}} u_x^2\, dx \int_{\mathbb{S}} u_x^2\, dx \end{aligned} \tag{3.18}$$

Combining (3.17) and (3.18) we obtain

$$\frac{d}{dt}\int_{\mathbb{S}} u_x^3\, dx \leq -\frac{3b-5}{2}\int_{\mathbb{S}} u_x^4\, dx + \left(\frac{3b}{8} - \frac{3}{4}\frac{3-b}{sh(\frac{1}{2})}\right)\left(\int_{\mathbb{S}} u_x^2\, dx\right)^2 \tag{3.19}$$

Case(i): $\frac{5}{3} < b \leq \frac{6}{2+sh(\frac{1}{2})}$

Since

$$\begin{aligned} &b \leq \frac{6}{2+sh(\frac{1}{2})} = 3bsh(\frac{1}{2}) \leq 18 - 6b = \frac{3b}{2} \leq \frac{3(3-b)}{sh(\frac{1}{2})} \\ &\implies \frac{3b}{2} \leq \frac{3(3-b)}{sh(\frac{1}{2})} = \frac{3b}{8} - \frac{3}{4}\frac{3-b}{sh(\frac{1}{2})} \leq 0 \end{aligned}$$

And this implies parenthesized term in the second expression is less than zero and therefore the term could be dropped altogether and we would have

$$\frac{d}{dt}\int_{\mathbb{S}} u_x^3\, dx \leq -\frac{3b-5}{2}\int_{\mathbb{S}} u_x^4\, dx. \tag{3.20}$$

If we define $V(t) := \int_{\mathbb{S}} u_x^3(t,x)\, dx$ for all $t \geq 0$, then

$$V(t) \leq V(0), \quad t \geq 0.$$

Since $V(0) < 0$, the above inequality implies that $V(t) < 0$ for all $t \geq 0$. It is then inferred that

$$\frac{d}{dt}V(t) \leq -\frac{3b-5}{2}\,(V(t))^{\frac{4}{3}}, \quad t > 0.$$

Thus we have

$$\left(\frac{(3b-5)t}{6} + \frac{1}{(V(0))^{\frac{1}{3}}}\right)^3 \leq \frac{1}{V(t)} < 0, \quad t \geq 0.$$

Since $V(0) < 0$, the above inequality will lead to a contradiction as $t \geq 0$ is big enough, which implies $T < \infty$.

Case(ii): $\dfrac{6}{2+sh(\frac{1}{2})} \leq b < 3$

Since

$$b \geq \frac{6}{2+sh(\frac{1}{2})} = 3bsh(\frac{1}{2}) \geq 18 - 6b = \frac{3b}{2} \geq \frac{3(3-b)}{sh(\frac{1}{2})}$$
$$= \frac{3b}{2} \geq \frac{3(3-b)}{sh(\frac{1}{2})} = \frac{3b}{8} - \frac{3}{4}\frac{3-b}{sh(\frac{1}{2})} \geq 0$$

Therefore this enables us to apply the Cauchy-Schwartz inequality and we consequently obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx \leq -\frac{3b-5}{2}\int_{\mathbb{S}} u_x^4\,dx + \left(\frac{3b}{8} - \frac{3}{4}\frac{3-b}{sh(\frac{1}{2})}\right)\left(\int_{\mathbb{S}} u_x^4\,dx\right) \\
&= -\left[\frac{3b-5}{2} + \frac{3b}{8} - \frac{3\,(3-b)}{4sh\left(\frac{1}{2}\right)}\right]\int_{\mathbb{S}} u_x^4\,dx \\
&= \left[\frac{-12b+20+3b}{8} - \frac{3\,(3-b)}{4sh\left(\frac{1}{2}\right)}\right]\int_{\mathbb{S}} u_x^4\,dx \qquad (3.21)\\
&= \left[\frac{-9b+20}{8} - \frac{3\,(3-b)}{4sh\left(\frac{1}{2}\right)}\right]\int_{\mathbb{S}} u_x^4\,dx \\
&= -\left[\frac{9b-20}{8} + \frac{3\,(3-b)}{4sh\left(\frac{1}{2}\right)}\right]\int_{\mathbb{S}} u_x^4\,dx
\end{aligned}$$

We claim that $\dfrac{9b-20}{8} + \dfrac{3\,(3-b)}{4sh\left(\frac{1}{2}\right)} > 0$ and hence we obtain another bound for b which is $b < \dfrac{18-20sh\left(\frac{1}{2}\right)}{6-9sh\left(\frac{1}{2}\right)}$.

But $3 < \dfrac{18-20sh\left(\frac{1}{2}\right)}{6-9sh\left(\frac{1}{2}\right)}$ and $b \leq 3$. This implies that $\dfrac{9b-20}{8} + \dfrac{3\,(3-b)}{4sh\left(\frac{1}{2}\right)} > 0$. Hence an application of Hölder's inequality on (3.21) yields

$$\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx \leq -K\left(\int_{\mathbb{S}} u_x^3\,dx\right)^{\frac{4}{3}}, \quad t \geq 0. \qquad (3.22)$$

where $K = \dfrac{9b-20}{8} + \dfrac{3\,(3-b)}{4sh\left(\frac{1}{2}\right)} > 0$.

If we define $V(t) := \int_{\mathbb{S}} u_x^3(t,x)\,dx$ for all $t \geq 0$, then

$$V(t) \leq V(0), \quad t \geq 0.$$

Since $V(0) \leq 0$, the above inequality implies that $V(t) < 0$ for all $t \geq 0$. It is then inferred that

$$\frac{d}{dt}V(t) \leq -\mathbb{K}\,(V(t))^{\frac{4}{3}}, \quad t > 0.$$

Thus we have

$$\left(\frac{Kt}{3} + \frac{1}{(V(0))^{\frac{1}{3}}}\right)^3 \leq \frac{1}{V(t)} < 0, \quad t \geq 0.$$

Since $V(0) < 0$, the above inequality will lead to a contradiction as $t \geq t_0$ is big enough, which implies $T < \infty$. ■

As immediate consequences of Theorem 3.4, we have

Corollary 3.6. If $u_0 \in H^3(\mathbb{S})$, $u_0 \not\equiv 0$ and $\int_{\mathbb{S}} u_0\,dx = 0$ or $\int_{\mathbb{S}} y_0\,dx = 0$, then the corresponding solution u to (5.1.1) blows up in finite time.

Proof. Note

$$\int_{\mathbb{S}} u(t,x)\,dx = \int_{\mathbb{S}} y(t,x)\,dx = \int_{\mathbb{S}} y_0(x)\,dx = \int_{\mathbb{S}} u_0(x)\,dx = 0.$$

The above relation shows that $u(t,x)$ has a zero for all $t \in \mathbb{S}$. It follows from the earlier proved theorem (3.4) that the solution u to (3.13) blows up in finite time. ■

4. Remarks

Therefore it is shown that if $m_0 = u_0 - u_{0xx}$ doesn't change sign, then $m(t), \forall t$ will not change sign as long as $m(t)$ exists. It is also proved that if u(t,x) is a solution to the quasi linear evolution equation with $u(0,x) = u_0(x)$, then $-u(t,-x)$ is also a solution to the same equation with initial profile $-u_0(-x)$. Due to the uniqueness of solutions, the solution to the b-family equation is odd as long as the initial profile $u_0(x)$ is odd. The blow-up theorem proved earlies speaks of this kind of data. Hence, we conclude that the solution to the b-family of equations blows up in finite time.

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