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Existence of Nonoscillatory Solutions of Even Order Nonlinear Neutral Difference Equations

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Abstract

In this paper, the existence of nonoscillatory solutions of the even order nonlinear neutral difference equations of the form

$$\Delta^{m-1}\big(r_n\Delta(x_n + p_n x_{n-k})\big) + q_n f(x_{n-l}) = h_n$$

are treated by using fixed point technique. Sufficient conditions for the existence of nonoscillatory solution of such equations are established. Examples are provided to illustrate the main results.

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1 Introduction

Consider the nonlinear neutral difference equation of the form

$$\Delta^{m-1}\big(r_n\Delta(x_n + p_n x_{n-k})\big) + q_n f(x_{n-l}) = h_n, n \in \mathbb{N}(n_0) \qquad (1.1)$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, k and l are positive integers, m is an even integer, $\{p_n\}, \{q_n\}$ and $\{h_n\}$ are real sequences defined for all $n \in \mathbb{N}(n_0) = \{n_0, n_{0+1}, n_{0+2}, \dots\}$, n_0 is a nonnegative integer and f is a continuous real valued function.

Let $\theta = max\,\{k\,, l\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \geq N(n_0 - \theta)$ and satisfying equation (1.1) for all $n \in \mathbb{N}_0$. A solution $\{x_n\}$ of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

The problem of the existence of nonoscillatory solution of nonlinear neutral difference equations received less attention as oscillation and nonoscillation problem. In this article, we apply the technique of Krasnoselskii's fixed point theorem to establish some sufficient conditions for the existence of nonoscillatory solutions of equation (1.1) without using nondecreasing conditions and any sign conditions on the sequences $\{q_n\}$ and $\{h_n\}$. Here we allow $\{q_n\}$ and $\{h_n\}$ to be oscillatory.

Lemma 1.1. (Krasnoselskii's Fixed Point Theorem)
Let X be a Banach space and let Ω be a bounded closed convex subset of X and S_1, S_2 be maps of Ω into X such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is a contractive and S_2 is completely continuous, then the equation $S_1x + S_2y = x$ has a solution in Ω.

Lemma 1.2. (Schauder's Fixed Point Theorem)
Let Ω be a closed, convex and nonempty subset of a Banach space X. Let S : Ω → Ω be a continuous mapping such that SΩ is relatively compact subset of X. Then S has atleast one fixed point in Ω. That is, there exists an $x \in \Omega$ such that $Sx = x$.

2 Existence of nonoscillatory solutions
In this section we establish sufficient conditions for the existence of bounded nonoscillatory solution of equation (1.1).

Theorem 2.1.
Assume that $-1 < c_1 \leq p_n \leq 0$ and that

$$\sum_{n=n_0}^{\infty} \frac{1}{r_n} \sum_{s=n}^{\infty} s^{(m-2)} \; |q_s| < \infty \tag{2.1}$$

and

$$\sum_{n=n_0}^{\infty} \frac{1}{r_n} \sum_{s=n}^{\infty} s^{(m-2)} \; |h_s| < \infty \tag{2.2}$$

where for $s, m \in \mathbb{N}(n_0)$

$$s^{(m)} = \prod_{i=0}^{m-1} (s-i) \; with \; s^{(0)} = 1.$$

Then equation (1.1) has a bounded nonoscillatory solution.

Proof.
By (2.1) and (2.2), we choose a $N \in \mathbb{N}(n_0)$ sufficiently large such that

$$\frac{1}{(m-2)!} \sum_{n=N}^{\infty} \frac{1}{r_n} \sum_{s=n}^{\infty} s^{(m-2)} \left(|q_s| M_1 + |h_s| \right) \leq \frac{(1+c_1)}{3}$$

where $M_1 = \max_{\frac{(1+c_1)}{3} \leq x \leq \frac{4}{3}} \{|f(x)|\}$. Let $B(n_0)$ be the set of all real sequence with the norm $\|x\| = \sup_{n\geq n_0}|x_n| < \infty$. Then $B(n_0)$ is a Banach space. We define a closed, bounded and convex subset Ω of $B(n_0)$ as follows,

$$\Omega = \left\{x = \{x_n\} \in B(n_0) : \frac{1}{3}(1 + c_1) \leq x_n \leq \frac{4}{3}, n \in \mathbb{N}(n_0)\right\}.$$

Define two maps S_1 and $S_2 : \Omega \to B(n_0)$ as follows,

$$(S_1x)_n = \begin{cases} 1 + c_1 - p_n x_{n-k}, n \geq N \\ (S_1x)_N, n_0 \leq n \leq N, \end{cases}$$

and

$$(S_2x)_n = \begin{cases} \frac{1}{(m-2)!}\sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}(j-s+m-2)^{(m-2)}q_j f(x_{j-l}) - h_j, n \geq N \\ (S_2x)_N, \quad n_0 \leq n \leq N. \end{cases}$$

(i) We shall show that for any $x, y \in \Omega, S_1x + S_2y \in \Omega$. Infact for every $x, y \in \Omega$ and $n \geq N$, we get

$$\begin{aligned}
&(S_1x)_n + (S_2y)_n \\
&\leq 1 + c_1 - p_n x_{n-k} + \frac{1}{(m-2)!}\sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}(j-s+m-2)^{(m-2)}\left(|q_j||y_{j-l}| + |h_j|\right) \\
&\leq 1 + c_1 - \frac{4}{3}c_1 + \frac{1}{(m-2)!}\sum_{s=N}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}s^{(m-2)}\left(|q_j|M_1 + |h_j|\right) \\
&\leq 1 + c_1 - \frac{4}{3}c_1 + \frac{1 + c_1}{3} = \frac{4}{3}.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&(S_1x)_n + (S_2y)_n \\
&\geq 1 + c_1 - p_n x_{n-k} - \frac{1}{(m-2)!}\sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}(j-s+m-2)^{(m-2)}\left(|q_j||y_{j-l}| + |h_j|\right) \\
&\geq 1 + c_1 - \frac{1}{(m-2)!}\sum_{s=N}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}s^{(m-2)}\left(|q_j|M_1 + |h_j|\right) \\
&\geq 1 + c_1 - \frac{1 + c_1}{3} = \frac{2(1 + c_1)}{3} \\
&\geq \frac{1 + c_1}{3}.
\end{aligned}$$

Hence

$$\frac{1 + c_1}{3} \leq (S_1x)_n + (S_2y)_n \leq \frac{4}{3} \; for \; n \geq N_0 .$$

Thus we have proved that

$$(S_1x)_n + (S_2y)_n \in \Omega \; for \; any \; x, y \in \Omega .$$

(ii) We shall show that S_1 is a contraction mapping on Ω. Infact for every $x, y \in \Omega$ and $n \geq N$ we have

$$|(S_1x)_n - (S_2y)_n| \leq -p_n|x_{n-k} - y_{n-k}|$$

$\leq -c_1\|x-y\|$.

Since $0 < -c_1 < 1$, we conclude that S_1 is a contraction mapping on Ω.

(iii)Next we show that S_2 is uniformly Cauchy. First we shall show that S is continuous. Let $\{x^{(i)}\}$ be a sequence in Ω such that $x^{(i)} \to x = \{x_n\}$ as $i \to \infty$. Since Ω is closed $x = \{x_n\} \in \Omega$.

Furthermore, for $n \geq N$ we have,

$$\left(S_2^{(i)}\, x\right)_n = \frac{1}{(m-2)!}\sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}\left[(j-s+m+2)^{(m-2)}q_j f\left(x_{j-l}^{(i)}\right)-h_j\right],$$

and

$$(S_2\, x)_n = \frac{1}{(m-2)!}\sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}\left[(j-s+m+2)^{(m-2)}q_j f(x_{j-l})-h_j\right].$$

Then

$$\left|\left(S_2^{(i)}\, x\right)_n-(S_2\, x)_n\right|$$
$$\leq \frac{1}{(m-2)!}\sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}(j-s+m+2)^{(m-2)}\left|q_j\right|\left|f\left(x_{j-l}^{(i)}\right)-f(x_{j-l})\right|$$
$$\leq \frac{1}{(m-2)!}\sum_{s=N}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}(j-s+m+2)^{(m-2)}\left|q_j\right|\left|f\left(x_{j-l}^{(i)}\right)-f(x_{j-l})\right|.$$

Since

$$\left|f\left(x_{j-l}^{(i)}\right)-f(x_{j-l})\right| \to 0 \; as\; i \to \infty,$$

we conclude that

$$\lim_{i\to\infty}\left\|\left(S_2^{(i)}\, x\right)_n-(S_2\, x)_n\right\| = 0.$$

This means that S_2 is continuous. Finally we prove that S_2 is uniformly Cauchy. By (2.1), for any $\in > 0$, choose $N_1 > N$ large enough so that

$$\frac{1}{(m-2)!}\sum_{n=N_1}^{\infty}\frac{1}{r_n}\sum_{s=n}^{\infty}s^{(m-2)}(|q_s|M_1+|h_s|) < \frac{\in}{2}.$$

Then for $x \in \Omega$, $n_2 > n_1 > N_1$

$$\left|(S_2\, x)_{n_2}-(S_2\, x)_{n_1}\right|$$
$$\leq \frac{1}{(m-2)!}\sum_{s=n_2}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}(j-s+m+2)^{(m-2)}\left(\left|q_j\right|\left|f(x_{j-l})\right|+\left|h_j\right|\right)$$
$$+\frac{1}{(m-2)!}\sum_{s=n_1}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}(j-s+m+2)^{(m-2)}\left(\left|q_j\right|\left|f(x_{j-l})\right|+\left|h_j\right|\right)$$
$$\leq \frac{1}{(m-2)!}\sum_{s=n_2}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}(j-s+m+2)^{(m-2)}\left(\left|q_j\right|M_1+\left|h_j\right|\right)$$

$$+\frac{1}{(m-2)!}\sum_{s=n_1}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}(j-s+m+2)^{(m-2)}\left(|q_j|M_1+|h_j|\right)$$
$$<\frac{\in}{2}+\frac{\in}{2}=\in.$$

Therefore $(S_2\,x)$ is uniformly Cauchy. By Lemma 1.2, there is an $x^* \in \Omega$ such that $S_1x^* + S_2x^* = x^*$. It is easy to see that $x^* = \{x_n^*\}$ is a nonoscillatory solution of the equation (1.1). This completes the proof of Theorem 2.1. ◘

Example 2.1.
Consider the difference equation

$$\Delta^3\left(n\Delta\left(x_n-\frac{1}{2}x_{n-1}\right)\right)+\frac{1}{n(n+1)(n+2)(n+3)(n+4)}\,x_{n-1}$$
$$=\frac{3n^2-26n-36}{(n-1)n(n+1)(n+2)(n+3)(n+4)}\quad n\geq 2\,.\ (2.3)$$

Here $r_n = n$, $p_n = -\frac{1}{2}$, $q_n = \frac{1}{n(n+1)(n+2)(n+3)(n+4)}$ and $h_n = \frac{3n^2-26n-36}{(n-1)n(n+1)(n+2)(n+3)(n+4)}$. It is easy to see that all conditions of Theorem 2.1 are satisfied and hence the equation (2.3) has a bounded nonoscillatory solution. Infact $\{x_n\} = \left\{1+\frac{1}{n}\right\}$ is one such solution of equation (2.3).

Theorem 2.2.
Assume that $-\infty < p_n \equiv c_2 < -1$ and that (2.1) and (2.2) hold. Then equation (1.1) has a bounded nonoscillatory solution.

Proof.
By (2.1) and (2.2), we choose a $N \in \mathbb{N}(n_0)$ sufficiently large such that

$$-\frac{1}{c_2}\frac{1}{(m-2)!}\sum_{n=N}^{\infty}\frac{1}{r_n}\sum_{s=n+k}^{\infty}s^{(m-2)}\left(|q_s|M_2+|h_s|\right)\leq-\frac{(c_2+1)}{2}$$

where $M_2 = \max_{-\frac{(c_2+1)}{2}\leq x\leq -2c_2}\{|f(x)|\}$.

Let $B(n_0)$ be the space defined as in the proof of Theorem 2.1. We define a closed, bounded and convex subset Ω of $B(n_0)$ as follows:

$$\Omega=\left\{x=\{x_n\}\in B(n_0):\ -\frac{(c_2+1)}{2}\leq x_n\leq -2c_2, n\in\mathbb{N}(n_0)\right\}.$$

Define two maps S_1 and $S_2 : \Omega \to B(n_0)$ as follows:

$$(S_1x)_n=\begin{cases}-c_2-1-\dfrac{1}{p_n}x_{n+k}\,, n\geq N\\ (S_1x)_N\,,\ n_0\leq n\leq N,\end{cases}$$

and

$$(S_2x)_n = \begin{cases} \frac{1}{p_n}\frac{1}{(m-2)!}\sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{j=s+k}^{\infty}(j-s-k+m-2)^{(m-2)}q_jf(x_{j-l}) - h_j, & n \geq N \\ (S_2x)_N, & n_0 \leq n \leq N. \end{cases}$$

We shall show that for any $x, y \in \Omega$, $S_1x + S_2y \in \Omega$. Infact for every $x, y \in \Omega$, we get

$$(S_1x)_n + (S_2y)_n$$
$$\leq -c_2 - 1 - \frac{1}{p_n}x_{n+k} - \frac{1}{p_n}\frac{1}{(m-2)!}\sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{j=s+k}^{\infty}(j-s-k+m-2)^{(m-2)}(|q_j||f(y_{j-l})| + |h_j|)$$
$$\leq -c_2 - 1 + 2 - \frac{1}{c_2}\frac{1}{(m-2)!}\sum_{s=N}^{\infty}\frac{1}{r_s}\sum_{j=s+k}^{\infty}s^{(m-2)}(|q_j|M_2 + |h_j|)$$
$$\leq c_2 - 1 + 2 - \frac{(c_2+1)}{2}$$
$$\leq -2c_2.$$

Furthermore, we have

$$(S_1x)_n + (S_2y)_n$$
$$\geq -c_2 - 1 - \frac{1}{p_n}x_{n+k} + \frac{1}{p_n}\frac{1}{(m-2)!}\sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{j=s+k}^{\infty}(j-s-k+m-2)^{(m-2)}(|q_j||f(y_{j-l})| + |h_j|)$$
$$\geq -c_2 - 1 + \frac{1}{c_2}\frac{1}{(m-2)!}\sum_{s=N}^{\infty}\frac{1}{r_s}\sum_{j=s+k}^{\infty}s^{(m-2)}(|q_j|M_2 + |h_j|)$$
$$\geq -c_2 - 1 + \frac{c_2-1}{2} = -\frac{(c_2+1)}{2}.$$

Hence

$$-\frac{c_2+1}{2} \leq (S_1x)_n + (S_2y)_n \leq -2c_2 \; for \; n \in \mathbb{N}(n_0).$$

Thus we have proved that

$S_1x + S_2y \in \Omega \; for \; any \; x, y \in \Omega$.

We shall show that S_1 is a contractive mapping on Ω. Infact, for $x, y \in \Omega$ and $n \geq N$ we have

$$|(S_1x)_n - (S_1y)_n| \leq -\frac{1}{p_n}|x_{n+k} - y_{n+k}|$$
$$\leq -\frac{1}{c_2}\|x - y\|.$$

Since $0 < -\frac{1}{c_2} < 1$, we conclude that S_1 is a contractive mapping on Ω.

Proceeding, similarly as in the proof of Theorem 2.1, we obtain S_2 is uniformly Cauchy. By Lemma 1.1, there is an $x^* \in \Omega$ such that $S_1x^* + S_2x^* = x^*$. Clearly, $x^* = \{x_n^*\}$ is a bounded nonoscillatory solution of the equation (1.1). This completes the proof of Theorem 2.2. ◻

Example 2.2.

Consider the difference equation

$$\Delta^3\left(2^n\Delta(x_n - 2x_{n-1})\right) + \frac{1}{2^n}\, x_{n-1} = \frac{1}{2^n}\left(1 + \frac{1}{2^{n-1}}\right), n \geq 1 \,. (2.4)$$

Here $r_n = 2^n, p_n = -2\,, q_n = \frac{1}{2^n}$ and $h_n = \frac{1}{2^n}\left(1 + \frac{1}{2^{n-1}}\right)$. It is easy to see that all conditions of Theorem 2.2 are satisfied and hence the equation (2.4) has a bounded nonoscillatory solution. Infact $\{x_n\} = \left\{1 + \frac{1}{2^n}\right\}$ is one such solution of equation (2.4).

Theorem 2.3.
Assume that $0 \leq p_n \leq c_3 < 1$ and that (2.1) and (2.2) hold. Then equation (1.1) has a bounded nonoscillatory solution.

Proof.
By (2.1) and (2.2), we choose a $N \in \mathbb{N}(n_0)$ sufficiently large such that

$$\frac{1}{(m-2)!}\sum_{n=N}^{\infty}\frac{1}{r_n}\sum_{s=n}^{\infty} s^{(m-2)}\left(|q_s|M_3 + |h_s|\right) \leq 1 - c_3$$

where $M_3 = \max_{2(1-c_3) \leq x \leq 4}\{|f(x)|\}$.
Let $B(n_0)$ be the space defined as in the proof of Theorem 2.1. We define a closed, bounded and convex subset Ω of $B(n_0)$ as follows:
$\Omega = \{x = \{x_n\} \in B(n_0) : 2(1 - c_3) \leq x_n \leq 4, n \in \mathbb{N}(n_0)\}$.
We define two maps S_1 and $S_2 : \Omega \to B(n_0)$ as follows:

$$(S_1x)_n = \begin{cases} 3 + c_3 - p_n x_{n-k}\,, n \geq N \\ (S_1x)_N\,,\ n_0 \leq n \leq N, \end{cases}$$

and

$$(S_2x)_n = \begin{cases} \dfrac{1}{(m-2)!}\displaystyle\sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}(j - s + m - 2)^{(m-2)}\left(q_j f(x_{j-l}) - h_j\right),\ n \geq N \\ (S_2x)_N\,, \qquad n_0 \leq n \leq N. \end{cases}$$

We shall show that for any $x, y \in \Omega$, $S_1x + S_2y \in \Omega$. Infact for every $x, y \in \Omega$, and $n \geq N$, we obtain

$$\begin{aligned}(S_1x)_n + (S_2y)_n &\leq 3 + c_3 - p_n x_{n-k} + \frac{1}{(m-2)!}\sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}(j - s + m - 2)^{(m-2)}\left(|q_j||f(y_{j-l})| + |h_j|\right)\\ &\leq 3 + c_3 + \frac{1}{(m-2)!}\sum_{s=N}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}s^{(m-2)}\left(|q_j|M_3 + |h_j|\right)\\ &\leq 3 + c_3 + 1 - c_3 = 4.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}(S_1x)_n + (S_2y)_n &\geq 3 + c_3 - p_n x_{n-k} - \frac{1}{(m-2)!}\sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}(j - s + m - 2)^{(m-2)}\left(|q_j||f(y_{j-l})| + |h_j|\right)\\ &\geq 3 + c_3 - 4c_3 - \frac{1}{(m-2)!}\sum_{s=N}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}s^{(m-2)}\left(|q_j|M_3 + |h_j|\right)\end{aligned}$$

$\geq 3 + c_3 - 4c_3 - (1 - c_3) = 2(1 - c_3)$.

Hence

$2(1 - c_3) \leq (S_1 x)_n + (S_2 y)_n \leq 4 , for\ n \geq n_0$.

Thus we have proved that

$(S_1 x) + (S_2 y) \in \Omega\ for\ any\ x, y \in \Omega$.

Proceeding, similarly as in the proof of Theorem 2.1, we obtain the mapping S_1 is a contractive on Ω and the mapping S_2 is uniformly Cauchy. By Lemma 1.1, there is an $x^* \in \Omega$ such that $S_1 x^* + S_2 x^* = x^*$. Clearly, $x^* = \{x_n^*\}$ is a bounded nonoscillatory solution of the equation (1.1). This completes the proof of Theorem 2.3. ◻

Example 2.3.

Consider the difference equation

$$\Delta^3\left((n+3)\Delta\left(x_n + \frac{1}{n+1}x_{n-1}\right)\right) + \frac{6(n+2)^3 x_{n-1}^3}{(n+1)^3(n+2)(n+3)(n+4)(n+5)} = \frac{6}{(n+4)(n+5)(n+6)(n+7)}, n \geq 2 . \quad (2.5)$$

Here $r_n = (n+3), p_n = \frac{1}{n+1}, q_n = \frac{6(n+2)^3 x_{n-1}^3}{(n+1)^3(n+2)(n+3)(n+4)(n+5)}$ and $h_n = \frac{6}{(n+4)(n+5)(n+6)(n+7)}$. It is easy to see that all conditions of Theorem 2.3 are satisfied and hence the equation (2.5) has a bounded nonoscillatory solution. Infact $\{x_n\} = \left\{\frac{n+2}{n+3}\right\}$ is one such solution of equation (2.5).

Theorem 2.4.

Assume that $1 < c_4 \equiv p_n < \infty$ and that (2.1) and (2.2) hold. Then equation (1.1) has a bounded nonoscillatory solution.

Proof.

By (2.1) and (2.2), we choose a $N \in \mathbb{N}(n_0)$ sufficiently large so that

$$\frac{1}{c_4}\frac{1}{(m-2)!}\sum_{n=N}^{\infty}\frac{1}{r_n}\sum_{s=n+k}^{\infty} s^{(m-2)}\left(|q_s|M_4 + |h_s|\right) < c_4 - 1$$

where $M_4 = \max_{2(c_4-1) \leq x \leq 4c_4} \{|f(x)|\}$.

Let $B(n_0)$ be the space defined as in the proof of Theorem 2.1. We define a closed, bounded and convex subset Ω of $B(n_0)$ as follows:

$$\Omega = \{x = \{x_n\} \in B(n_0) : 2(c_4 - 1) \leq x_n \leq 4c_4 , n \in \mathbb{N}(n_0)\}.$$

Define two maps S_1 and $S_2 : \Omega \to B(n_0)$ as follows:

$$(S_1 x)_n = \begin{cases} 3c_4 + 1 - \frac{1}{p_n} x_{n+k} , n \geq N \\ (S_1 x)_N , n_0 \leq n \leq N, \end{cases}$$

and

$$(S_2x)_n = \begin{cases} \dfrac{1}{(m-2)!\,p_n}\displaystyle\sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{j=s+k}^{\infty}(j-s-k+m-2)^{(m-2)}\left(q_j f(x_{j-l}) + h_j\right), & n \geq N \\ (S_2x)_N, & n_0 \leq n \leq N. \end{cases}$$

We shall show that for any $x, y \in \Omega$, $S_1x + S_2y \in \Omega$. Infact for every $x, y \in \Omega$ and $n \geq N$, we obtain

$(S_1x)_n + (S_2y)_n$

$$\leq 3c_4 + 1 - \frac{1}{p_n}x_{n+k} + \frac{1}{(m-2)!\,p_n}\sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{j=s+k}^{\infty}(j-s-k+m-2)^{(m-2)}\left(|q_j||f(y_{j-l})| + |h_j|\right)$$

$$\leq 3c_4 + 1 + \frac{1}{(m-2)!\,c_4}\sum_{s=N}^{\infty}\frac{1}{r_s}\sum_{j=s+k}^{\infty}s^{(m-2)}\left(|q_j|M_4 + |h_j|\right)$$

$\leq 3c_4 + 1 + c_4 - 1 = 4c_4$.

Furthermore, we have

$(S_1x)_n + (S_2y)_n$

$$\geq 3c_4 + 1 - \frac{1}{p_n}x_{n+k} - \frac{1}{(m-2)!\,p_n}\sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{j=s+k}^{\infty}(j-s-k+m-2)^{(m-2)}\left(|q_j||f(y_{j-l})| + |h_j|\right)$$

$$\geq 3c_4 + 1 - 4 - \frac{1}{(m-2)!\,c_4}\sum_{s=N}^{\infty}\frac{1}{r_s}\sum_{j=s+k}^{\infty}s^{(m-2)}\left(|q_j|M_4 + |h_j|\right)$$

$$\geq 3c_4 - 3 - (c_4 - 1) = 2(c_4 - 1).$$

Hence

$2(c_4 - 1) \leq (S_1x)_n + (S_2y)_n \leq 4c_4$, for $n \in \mathbb{N}(n_0)$.

Thus we have proved that

$(S_1x) + (S_2y) \in \Omega$ for any $x, y \in \Omega$.

Proceeding, similarly as in the proof of Theorem 2.1, we obtain the mapping S_1 is a contractive on Ω and the mapping S_2 is uniformly Cauchy. By Lemma 1.1, there is an $x^* \in \Omega$ such that $S_1x^* + S_2x^* = x^*$. Clearly, $x^* = \{x_n^*\}$ is a bounded nonoscillatory solution of the equation (1.1). This completes the proof of Theorem 2.4. ◘

Example 2.4.

Consider the difference equation

$$\Delta^3\left(3^n\Delta(x_n + 2x_{n-1})\right) + \frac{1}{3^n}x_{n-1} = \frac{1}{3^n}\left(1 + \frac{1}{3^{n-1}}\right), n \geq 2. \quad (2.6)$$

Here $r_n = 3^n, p_n = 2, q_n = \frac{1}{3^n}$ and $h_n = \frac{1}{3^n}\left(1 + \frac{1}{3^{n-1}}\right)$. It is easy to see that all conditions of Theorem 2.4 are satisfied and hence the equation (2.6) has a bounded nonoscillatory solution. Infact $\{x_n\} = \left\{1 + \frac{1}{3^n}\right\}$ is one such solution of equation (2.6).

Theorem 2.5.

Assume that $p_n \equiv 1$ and that (2.1) and (2.2) hold. Then equation (1.1) has a bounded nonoscillatory solution.

Proof.

By (2.1) and (2.2), we choose a $N > n_0$ sufficiently large such that

$$\frac{1}{(m-2)!}\sum_{n=N}^{\infty}\frac{1}{r_n}\sum_{s=n+k}^{\infty} s^{(m-2)}\left(|q_s|M_5+|h_s|\right)\leq 1$$

where $M_5 = \max_{2\leq x\leq 4}\{|f(x)|\}$.

We define a closed, bounded and convex subset Ω of $B(n_0)$ as follows:

$\Omega = \{x = \{x_n\}\in B(n_0) : 2 \leq x_n \leq 4, n\in\mathbb{N}(n_0)\}$.

Define a map $S : \Omega \to B(n_0)$ as follows:

$$(Sx)_n = \begin{cases} 3+\dfrac{1}{(m-2)!}\displaystyle\sum_{n=N}^{\infty}\frac{1}{r_n}\sum_{j=1}^{\infty}\sum_{s=n+(2j-1)k}^{n+2jk}(s-n+m-2)^{(m-2)}(q_s f(x_{s-l})+h_s), & n\geq N\\ (Sx)_N, & n_0\leq n\leq N.\end{cases}$$

We shall show that for any $S\Omega\subset\Omega$ for every $x\in\Omega$ and $n\geq N$, we get

$$(Sx)_n \leq 3+\frac{1}{(m-2)!}\sum_{n=N}^{\infty}\frac{1}{r_n}\sum_{j=1}^{\infty}\sum_{s=n+(2j-1)k}^{n+2jk}(s-n+m-2)^{(m-2)}(|q_s||f(x_{s-l})|+|h_s|)$$

$$\leq 3+\frac{1}{(m-2)!}\sum_{n=N}^{\infty}\frac{1}{r_n}\sum_{j=1}^{\infty}\sum_{s=n+(2j-1)k}^{n+2jk}s^{(m-2)}(|q_s|M_5+|h_s|)$$

$$\leq 4.$$

Furthermore, we have

$$(Sx)_n \geq 3-\frac{1}{(m-2)!}\sum_{n=N}^{\infty}\frac{1}{r_n}\sum_{j=1}^{\infty}\sum_{s=n+(2j-1)k}^{n+2jk}(s-n+m-2)^{(m-2)}(|q_s||f(x_{s-l})|+|h_s|)$$

$$\geq 3-\frac{1}{(m-2)!}\sum_{n=N}^{\infty}\frac{1}{r_n}\sum_{j=1}^{\infty}\sum_{s=n+(2j-1)k}^{n+2jk}s^{(m-2)}(|q_s|M_5+|h_s|)$$

$$\geq 2.$$

Hence, $S\Omega\subset\Omega$.

Proceeding, similarly as in the proof of Theorem 2.1, we obtain the mapping S is uniformly Cauchy. By Lemma 1.1, there is an $x^*\in\Omega$ such that $Sx^* = x^*$, that is

$$x_n^* = \begin{cases} 3+\dfrac{1}{(m-2)!}\displaystyle\sum_{n=N}^{\infty}\frac{1}{r_n}\sum_{j=1}^{\infty}\sum_{s=n+(2j-1)k}^{n+2jk}(s-n+m-2)^{(m-2)}(q_s f(x_{s-l})-h_s), & n\geq N\\ x_N^*, & n_0\leq n\leq N.\end{cases}$$

It follows that

$$x_n + x_{n-k} = 6+\frac{1}{(m-2)!}\sum_{s=n}^{\infty}\frac{1}{r_s}\sum_{j=s}^{\infty}(j-s+m-2)^{(m-2)}(q_s f(x_{s-l})-h_s).$$

Clearly, $x^* = \{x_n^*\}$ is a bounded nonoscillatory solution of the equation (1.1). This completes the proof of Theorem 2.5. ◻

Example 2.5.

Consider the difference equation

$$\Delta^3\left(2^n\Delta(x_n+x_{n-1})\right)+\frac{6}{n(n+1)(n+2)}x_{n-1} = \frac{6(2^n+2)}{2^n n(n+1)(n+2)}, n\geq 2. \quad (2.7)$$

Here $r_n = 2^n, p_n = 1, q_n = \frac{6}{n(n+1)(n+2)}$ and $h_n = \frac{6(2^n+2)}{2^n n(n+1)(n+2)}$. It is easy to see that all conditions of Theorem 2.5 are satisfied and hence the equation (2.7) has a bounded nonoscillatory solution. Infact $\{x_n\} = \left\{1 + \frac{1}{2^n}\right\}$ is one such solution of equation (2.7).

Theorem 2.6.
Assume that $p_n \equiv -1$ and that

$$\sum_{n=n_0}^{\infty} \frac{1}{r_n} \sum_{s=n}^{\infty} s^{(m-1)} \ |q_s| < \infty \quad (2.8)$$

and

$$\sum_{n=n_0}^{\infty} \frac{1}{r_n} \sum_{s=n}^{\infty} s^{(m-1)} \ |h_s| < \infty\,. \quad (2.9)$$

Then equation (1.1) has a bounded nonoscillatory solution.

Proof.
First note that the assumptions (2.8) and (2.9) are equivalent to

$$\sum_{n=n_0}^{\infty} \frac{1}{r_n} \sum_{j=0}^{\infty} \sum_{s=n+jk}^{\infty} s^{(m-2)} \ |q_s| < \infty, \qquad (2.10)$$

and

$$\sum_{n=n_0}^{\infty} \frac{1}{r_n} \sum_{j=0}^{\infty} \sum_{s=n+jk}^{\infty} s^{(m-2)} \ |h_s| < \infty \quad (2.11)$$

respectively. We choose a sufficiently large $N \in \mathbb{N}(n_0)$ such that

$$\frac{1}{(m-2)!} \sum_{n=N}^{\infty} \frac{1}{r_n} \sum_{j=1}^{\infty} \sum_{s=n+jk}^{\infty} s^{(m-2)} \left(|q_s| M_6 + |h_s| \right) \le 1$$

where $M_6 = \max_{0 \le x \le 1} \{|f(x)|\}$.
We define a closed, bounded and convex subset Ω of $B(n_0)$ as follows,
$\Omega = \{x = \{x_n\} \in B(n_0) : 2 \le x_n \le 4, n \in \mathbb{N}(n_0)\}$.
Define a map $S : \Omega \to B(n_0)$ as follows:

$$(Sx)_n = \begin{cases} 3 - \dfrac{1}{(m-2)!} \displaystyle\sum_{n=n_0}^{\infty} \frac{1}{r_n} \sum_{j=1}^{\infty} \sum_{s=n+jk}^{\infty} (s-n+m-2)^{(m-2)} (q_s f(x_{s-l}) + h_s)\,, & n \ge N \\ (Sx)_N\,, & n_0 \le n \le N. \end{cases}$$

We shall show that for any $S\Omega \subset \Omega$. Infact for every $x \in \Omega$ and $n \ge N$, we get

$$(Sx)_n \le 3 + \frac{1}{(m-2)!} \sum_{n=n_0}^{\infty} \frac{1}{r_n} \sum_{j=1}^{\infty} \sum_{s=n+jk}^{\infty} (s-n+m-2)^{(m-2)} (|q_s||f(x_{s-l})| + |h_s|)$$

$$\le 3 + \frac{1}{(m-2)!} \sum_{n=N}^{\infty} \frac{1}{r_n} \sum_{j=1}^{\infty} \sum_{s=n+jk}^{\infty} s^{(m-2)} (|q_s| M_6 + |h_s|)$$

≥ 2 .
Hence, $S\Omega \subset \Omega$. We now show that S is continuous.
Let $\{x^{(i)}\}$ be a sequence in Ω such that $x^{(i)} \to x = \{x_n\}$ as $i \to \infty$.

Since Ω is closed, $x = \{x_n\} \in \Omega$. Furthermore, for $n \geq N$ we have,

$$\left|\left(S^{(i)}\,x\right)_n - (S\,x)_n\right| \leq \frac{1}{(m-2)!}\sum_{n=n_0}^{\infty}\frac{1}{r_n}\sum_{j=1}^{\infty}\sum_{s=n+jk}^{\infty} s^{(m-2)}|q_s|\left|f\left(x_{s-l}^{(i)}\right) - f(x_{s-l})\right|.$$

Since

$$\left|f\left(x_{s-l}^{(i)}\right) - f(x_{s-l})\right| \to 0 \; as \; i \to \infty,$$

we conclude that

$$\lim_{i\to\infty}\left\|\left(S^{(i)}\,x\right)_n - (S\,x)_n\right\| = 0.$$

This means that S is continuous. Now, we show that S is uniformly Cauchy. By (2.10) and (2.11), for any $\in > 0$, choose $N_1 > N$ large enough so that

$$\frac{1}{(m-2)!}\sum_{n=N_1}^{\infty}\frac{1}{r_n}\sum_{j=1}^{\infty}\sum_{s=N_1+jk}^{\infty} s^{(m-2)}(|q_s|M_6 + |h_s|) < \frac{\in}{2}.$$

Then for $x \in \Omega$, $n_2 \geq n_1 \geq N$

$$\begin{aligned}\left|(S\,x)_{n_2} - (S\,x)_{n_1}\right| &\leq \frac{1}{(m-2)!}\sum_{n=n_2}^{\infty}\frac{1}{r_n}\sum_{j=1}^{\infty}\sum_{s=n+jk}^{\infty} s^{(m-2)}(|q_s||f(x_{s-l})| + |h_s|)\\ &+ \frac{1}{(m-2)!}\sum_{n=n_1}^{\infty}\frac{1}{r_n}\sum_{j=1}^{\infty}\sum_{s=n+jk}^{\infty} s^{(m-2)}(|q_s||f(x_{s-l})| + |h_s|)\\ &\leq \frac{1}{(m-2)!}\sum_{n=n_2}^{\infty}\frac{1}{r_n}\sum_{j=1}^{\infty}\sum_{s=n+jk}^{\infty} s^{(m-2)}(|q_s|M_6 + |h_s|)\\ &+ \frac{1}{(m-2)!}\sum_{n=n_1}^{\infty}\frac{1}{r_n}\sum_{j=1}^{\infty}\sum_{s=n+jk}^{\infty} s^{(m-2)}(|q_s|M_6 + |h_s|)\\ &< \frac{\in}{2} + \frac{\in}{2} = \in.\end{aligned}$$

Therefore Sx is uniformly Cauchy. By Lemma 1.2, there is an $x^* \in \Omega$ such that $Sx^* = x^*$. That is

$$x_n^* = \begin{cases} 3 - \dfrac{1}{(m-2)!}\displaystyle\sum_{n=n_0}^{\infty}\frac{1}{r_n}\sum_{j=1}^{\infty}\sum_{s=n+jk}^{\infty}(s-n+m-2)^{(m-2)}(q_s f(x_{s-l}^*) - h_s)\,, & n \geq N \\ x_N^*\,, & n_0 \leq n \leq N. \end{cases}$$

It follows that

$$x_n - x_{n-k} = -\frac{1}{(m-2)!}\sum_{n=n_0}^{\infty}\frac{1}{r_n}\sum_{s=n}^{\infty}(s-n+m-2)^{(m-2)}(q_s f(x_{s-l}) - h_s)\,, n \geq N.$$

Clearly, $x^* = \{x_n^*\}$ is a bounded nonoscillatory solution of the equation (1.1). This completes the proof of Theorem 2.6. ◻

Example 2.6.
Consider the difference equation

$$\Delta^3\big(3^n\Delta(x_n - x_{n-2})\big) + \frac{24}{n(n+1)}\, x_{n-2} = \frac{24(3^n+9)}{3^n n(n+1)}, n \geq 2 . \quad (2.12)$$

Here $r_n = 3^n$, $p_n = -1$, $q_n = \frac{24}{n(n+1)}$ and $h_n = \frac{24(3^n+9)}{3^n n(n+1)}$. It is easy to see that all conditions of Theorem 2.6 are satisfied and hence the equation (2.12) has a bounded nonoscillatory solution. Infact $\{x_n\} = \left\{1 + \frac{1}{3^n}\right\}$ is one such solution of equation (2.12).

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