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Oscillation Solutions to Third Order Half-linear Neutral Differential Equations with "Maxima"

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Abstract

In this paper, we study the oscillation of solutions to third-order half-linear neutral differential equations with "maxima"

$$\left(a(t)\left(\left(x(t)+p(t)x(\sigma(t))\right)''\right)^{\alpha}\right)' + q(t)\max_{[\tau(t),t]} x^{\alpha}(s) = 0, t \geq t_0 \geq 0$$

whereα is the ratio of odd positive integers and $\int_{t_0}^{t} a^{-\frac{1}{\alpha}}(t)dt = \infty$. Examples are given
to illustrate the main results.

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Keywords and Phrases: Oscillation, half-linear, neutral, third order, differential equations with "maxima".

1. Introduction

In this paper, we study the oscillation of solutions to the third-order differential equations with "maxima" of the form.

$$\left(a(t)\left(\left(x(t)+p(t)x(\sigma(t))\right)''\right)^{\alpha}\right)' + q(t)\max_{[\tau(t),t]} x^{\alpha}(s) = 0, t \geq t_0 \geq 0. \tag{1.1}$$

We assume the following conditions:

(C_1) $a(t), p(t), q(t), \tau(t)$ *and* $\sigma(t)$ *are in* $C([0,\infty])$;

$a(t), p(t), q(t), \tau(t)$ and $\sigma(t)$are positive functions; α is the quotient of odd positive integers;

(C_2)there is a constant p such that $0 \leq p(t) \leq p < 1$; the delay arguments satisfy

$$\tau(t) \leq t, \sigma(t) \leq t, lim_{t\to\infty}\tau(t) = lim_{t\to\infty}\sigma(t) = \infty;$$

(C_3) $a(t)$is positive and nonincreasing for all$t \geq t_0$and$\int_{t_0}^{t} a^{-\frac{1}{\alpha}}(s)ds \to \infty$ as $t \to \infty$.

By a solution of equation (1.1) we mean a continuous function $x(t) \in C^2([T_x, \infty)), T_x \geq t_0$, which has the property $\left(a(t)\left(z(t)\right)''\right)^{\alpha}$ are continuously differentiable and $x(t)$ satisfies the equation (1.1) on $[T_x, \infty)$ and $z(t) = \left(x(t) + p(t)x\left(\sigma(t)\right)\right)$. We consider only those solutions $x(t)$ of equation (1.1) which satisfy $\sup\{|x(t)|: t \geq T_x\} > 0$ for all $t \geq T_x$. A solution $x(t)$ of equation (1.1) is called oscillatory if it has arbitrary large zeros $[T_x, \infty)$, otherwise it is called nonoscillatory.

In the last few years, the qualitative theory of differential equations with "maxima" received very little attention even though such equations often arise in the problem of automatic regulation of various real system, see for example [2, 15]. The oscillatory behavior of solutions of differential equations with "maxima" are discussed in [3-7, 12, 13], and the references cited therein.

In [8], the authors obtained some sufficient conditions for the equation of the type (1.1) without maxima. In this paper, we extend the results of [1] to equation with maxima. Here we follow the same strategy as in [1], but with new estimates in Lemma 2.3, 2.4 and 2.5. Motivated by this observation, in this paper, we present some sufficient conditions for the oscillation of all solutions of equation (1.1).

In Section 2, we obtain criteria for the oscillation of allsolutions of equation (1.1) and in Section 3 we present someexamples to illustrate the main results.

Remark 1.1. All functional inequalities consider in this paper assumed to hold eventually, that is they are satisfied for all t large enough.

Remark 1.2. Without loss of generality we can deal only with the positive solution of equation (1.1), since the proof for the negative solution is similar.

2. Oscillation Results

In this section, we obtain a oscillatory criterion for equation (1.1). For a solution $x(t)$ of (1.1) we define the corresponding function $z(t)$ by

$$z(t) = x(t) + p(t)x\left(\sigma(t)\right). \tag{2.1}$$

To obtain sufficient condition for the oscillation of solutions of equation (1.1), we need the following lemmas.

Lemma 2.1. Let $x(t)$be a positive solution of equation (1.1), then there are only the following two cases for $z(t)$ defined in (2.1) hold:

(I)$z(t) > 0, z'(t) > 0$ *and* $z''(t) > 0$;

(II)$z(t) > 0, z'(t) < 0$ *and* $z''(t) > 0$ *for* $t \geq t_1 \geq t_0$;

where t_1 is sufficiently large.

The proof of the lemma is found in[8].

Lemma 2.2. Let $x(t)$ be a positive solution of equation (1.1) and let the corresponding $z(t)$ satisfy Lemma 2.1 (II). If

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\left(\frac{1}{a(u)}\int_{u}^{\infty}q(s)ds\right)^{1/\alpha}dudv = \infty \tag{2.2}$$

then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0.$

Proof. The proof is similar to that of in [8] and hence the details are omitted.

Lemma 2.3 Assume that $u(t) > 0, u'(t) \geq 0, u''(t) \leq 0,$ on $[t_0, \infty)$. Then for each $l \in (0,1)$ there exists $a\ T_l \geq t_0$ such that

$$\frac{u(\tau(t))}{A(\tau(t))} \geq l\frac{u(t)}{A(t)} \quad for\ t \geq T_l.$$

Lemma 2.4. Assume that $z(t) > 0, z'(t) > 0, z''(t) > 0, z'''(t) \leq 0, on\ [T_l, \infty)$. Then

$$\frac{z(t)}{z'(t)} \geq \frac{a^{1/\alpha}(t)A(t)}{2} \quad for\ t \geq T_l.$$

The proofs of Lemma 2.3 and Lemma 2.4 are found in [8].

Lemma 2.5. The function $x(t)$ is a negative solutions of equation (1.1) if and only if $-x(t)$ is a positive solution of the equation

$$\left(a(t)\left(\left(x(t)+p(t)x(\sigma(t))\right)''\right)^{\alpha}\right)' + q(t)\min_{[\tau(t),t]} x^{\beta}(s) = 0. \tag{2.3}$$

Proof. The assertion can be verified easily.

Lemm0a 2.6 Assume that

$z'(t) > 0, z''(t) > 0, (a(t)(z''(t))^{\alpha})' \leq 0$ on $[T_l, \infty)$.Then

$$\frac{A(t)z''(t)}{A'(t)z'(t)} \leq 1, \quad for\ t \geq T_l.$$

Proof. The proof is found in[8].

Now, we present the main results. For simplicity we introducethe following notations:

$$P = \lim_{t\to\infty} inf\ A^{\alpha}(t)\int_{t}^{\infty} P_l(s)ds,$$

$$Q = \lim_{t\to\infty} sup\ \frac{1}{A(t)}\int_{t_0}^{t} A^{\alpha+1}(s)P_l(s)ds$$

where

$$P_l(s) = l^{\propto} \max_{[\tau(t),t]} \left(1-p(s)\right)^{\alpha} q(s) a\left(\tau(t)\right) \left(\frac{A(\tau(s))}{A(s)}\right)^{\alpha} \left(\frac{A(\tau(s))}{2}\right)^{\alpha} \tag{2.4}$$

with $l \in (0,1)$ arbitrarily chosen and T_l large enough. Moreover for $z(t)$ satisfying case (I) of Lemma 2.1, we define

$$w(t) = a(t)\left(\frac{z''(t)}{z(t)}\right)^{\alpha}, \tag{2.5}$$

$$r = \lim_{t\to\infty} \inf A^{\propto}(t) w(t),$$

and

$$R = \lim_{t\to\infty} \sup A^{\alpha}(t) w(t). \tag{2.6}$$

Theorem 2.1. Assume that condition (2.2) holds. If

$$P = \lim_{t\to\infty} \inf A^{\propto}(t) \int_t^{\infty} P_l(s) ds > \frac{\propto^{\propto}}{(\propto+1)^{\propto+1}}, \tag{2.7}$$

then $x(t)$ is either oscillatory or $lim_{t\to\infty} x(t) = 0$.

Proof. Assume that $x(t)$ is a nonoscillatory solution of equation(1.1). We can assume without loss of generality that $x(t)$is positiveand the corresponding function $z(t)$ satises Case(I) of Lemma 2.1.

First note that

$$x(t) = z(t) - p(t)x\left(\sigma(t)\right) \geq \left(1-p(t)\right)z(t) \tag{2.8}$$

or

$$\max_{[\tau(t),t]} x^{\propto}(s) \geq z^{\propto}(t) \max_{[\tau(t),t]} \left(1-p(s)\right)^{\alpha}.$$

Using the above inequality in equation(1.1) we obtain

$$\left(a(t)\left(z''(t)\right)^{\alpha}\right)' + q(t) \max_{[\sigma(t),t]} \left(1-p(s)\right)^{\alpha} z^{\alpha}(t) \leq 0. \tag{2.9}$$

From the definition of $w(t)$ we see that$w(t) > 0$ and from (1.1) wehave

$$w'(t) \leq \frac{-q(t) z^{\alpha}\left(\tau(t)\right) \max_{[\tau(t),t]} \left(1-p(s)\right)^{\alpha}}{(z'(t))^{\alpha}} - \frac{\propto}{a^{1/\propto}(t)} w^{\frac{\propto+1}{\propto}}(t). \tag{2.10}$$

From Lemma 2.3 with $u(t) = z'(t)$,we have,

$$\frac{1}{z'(t)} \geq l \frac{A\left(\tau(t)\right)}{A(t)} \frac{1}{z'\left(\tau(t)\right)}, \quad t \geq T_l$$

where, l is the same as in $P_l(t)$. Now (2.10) becomes

$$w'(t) \leq -q(t) l^{\propto} \left(\frac{A\left(\tau(t)\right)}{A(t)}\right)^{\propto} \frac{z^{\propto}\left(\tau(t)\right)}{\left(z^{\alpha}\left(\tau(t)\right)\right)^{\alpha}} \max_{[\tau(s),s]} \left(1-p(s)\right)^{\propto}$$

$$-\frac{\propto}{a^{1/\propto}(t)}w^{\frac{\alpha+1}{\alpha}}(t).$$

Using the fact from Lemma 2.4 that $z(t) \geq \frac{a^{1/\alpha}A(t)}{2}z'(t)$, we have

$$w'(t) + P_l(t) + \frac{\propto}{a^{1/\propto}(t)}w^{\frac{\alpha+1}{\alpha}}(t) \leq 0 \tag{2.11}$$

Since $P_l(t) > 0$ *and* $w(t) > 0$ *for* $t \geq T_l$, it follows that $w'(t) \leq 0$ and

$$-\left(\frac{w'(t)}{\alpha w^{\frac{\alpha+1}{\alpha}}(t)}\right) > \frac{1}{a^{1/\propto}(t)},\ for\ t \geq T_l \tag{2.12}$$

This implies that

$$\left(\frac{1}{w^{1/\propto}(t)}\right)' > \frac{1}{a^{1/\propto}(t)}. \tag{2.13}$$

Integrating the last inequality from T_l *to* t,and using that $w^{-1/\alpha}(T_l) > 0$, we obtain

$$w(t) < \frac{1}{\left(\int_{T_l}^{t} a^{-\frac{1}{\propto}}(s)ds\right)^{\propto}}, \tag{2.14}$$

whichinview of (C_3) implies that $lim_{t\to\infty}w(t) = 0$. On theotherhand, from the definition of $w(t)$, and Lemma 2.5, we seethat

$$A^{\propto}(t)w(t) = a(t)\left(\frac{A(t)z''(t)}{z'(t)}\right)^{\propto} = \left(\frac{A(t)z''(t)}{A'(t)z'(t)}\right)^{\propto} \leq 1^{\propto}. \tag{2.15}$$

Then

$$0 \leq r \leq R \leq 1. \tag{2.16}$$

Now, let $\varepsilon > 0$, then from the definitions of P and r we can pick $t_2 \in [T_l, \infty)$sufficiently large that

$$A^{\alpha}\int_t^{\infty} P_l(s)ds \geq P - \varepsilon,$$

and

$$A^{\alpha}(t)w(t) \geq r - \varepsilon,\ for\ t \geq t_2,$$

Integrating (2.11) from t *to* ∞ and using $lim_{t\to\infty}w(t) = 0$, we have

$$w(t) \geq \int_t^{\infty} P_l(s)ds + \propto \int_t^{\infty}\frac{w^{1+1/\alpha}(s)}{a^{1/\alpha}(s)}ds, \qquad for\ t \geq t_2. \tag{2.17}$$

Assume $P = \infty$, then from (2.17), we have

$$A^{\propto}(t)w(t) \geq A^{\alpha}(t)\int_t^{\infty} P_l(s)ds.$$

Taking the limit infimum on both sides at $t \to \infty$, we get in view of (2.16) that $1 \geq r \geq \infty$. This is a contradiction. Next assume that $P < \infty$. Now from (2.17) and the fact $a'(t) \leq 0$, we have

$$A^{\alpha}(t)\,w(t) \geq A^{\alpha}(t)\int_t^{\infty} P_l(s)ds + \alpha A^{\alpha}(t)\int_t^{\infty}\frac{A^{\alpha+1}(s)w^{\frac{\alpha+1}{\alpha}}(s)}{A^{\alpha+1}(s)a^{1/\alpha}(s)}ds$$
$$\geq (P - \ ) + (r - \ )^{1+\frac{1}{\alpha}}A^{\alpha}\int_t^{\infty}\frac{\alpha A'(s)}{A^{\alpha+1}(s)}ds$$

and so

$$A^{\alpha}(t)w(t) \geq (P - \ ) + (r - \ )^{1+\frac{1}{\alpha.}}$$

Taking the limit infimum on both sides at $t \to \infty$, we get,

$$r \geq (P - \ ) + (r - \ )^{1+\frac{1}{\alpha.}}.$$

Since $\ > 0$ is arbitrary, we obtain

$$P \leq r - r^{1+\frac{1}{\alpha.}}.$$

Using the inequality,

$$Bu - Du^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}$$

with $B = D = 1$, we get

$$P \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}$$

which Contradicts (2.7). This completes the proof.
From Theorem 2.1 we have the following corollary.

Corollary 2.1. Assume that (2.2) holds. if

$$\lim_{t\to\infty}\inf A^{\alpha}(t)\int_t^{\infty} q(s)\max_{[\tau(t),t]}\left(1-p(s)\right)^{\alpha} a\left(\tau(s)\right)\frac{A\left(\tau(s)\right)^{2\alpha}}{A^{\alpha}}ds$$
$$> \frac{(2\alpha)^{\alpha}}{l^{\alpha}}\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}. \tag{2.18}$$

then every solution of equation (1.1) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Theorem 2.2.Assume that condition (2.2)holds. If

$$P + Q > 1, \tag{2.19}$$

then every solution of equation (1.1) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose $x(t)$ to be a nonoscillatory solution of equation (1.1) and since $-x(t)$ is also a solution, we can assume without loss of generality that $x(t)$is positive. Let the corresponding function $z(t)$ satisfies case(I) of Lemma 2.1.

Proceeding as in the proof of Theorem 2.1, we obtain (2.11). Now multiply (2.11) by $A^{\alpha+1}(t)$,and integrating from t_2 to t $(t \geq t_2)$, we get

$$\int_{t_2}^{t} A^{\alpha+1}(s)w'(s)ds \leq -\int_{t_2}^{t} A^{\alpha+1}(s)P_l(s)ds$$

$$-\alpha \int_{t_2}^{t} \frac{\left(A^{\alpha}(s)w(s)\right)^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}(s)}\, ds. \quad (2.20)$$

Using integration by parts, we obtain

$$A^{\alpha+1}(t)w(t) \leq A^{\alpha+1}(t_2)w(t_2) - \int_{t_2}^{t} A^{\alpha+1}(s)P_l(s)ds$$

$$-\alpha \int_{t_2}^{t} \frac{\left(A^{\alpha}(s)w(s)\right)^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}(s)}\, ds + \int_{t_2}^{t} w(s)\left(A^{\alpha+1}(s)\right)' ds.$$

Hence

$$A^{\alpha+1}(t)w(t) \leq A^{\alpha+1}(t_2)w(t_2) - \int_{t_2}^{t} A^{\alpha+1}(s)P_l(s)ds$$

$$+ \int_{t_2}^{t} \left[\frac{(\alpha+1)A^{\alpha}(s)w(s)}{a^{1/\alpha}(s)} - \frac{\alpha\left(A^{\alpha}(s)w(s)\right)^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}(s)}\right] ds.$$

Using the inequality

$$Bu - Du^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} . \frac{B^{\alpha+1}}{D^{\alpha}},$$

with

$$u(t) = A^{\alpha}(t)w(t),$$

and positive constants,

$$D = \frac{\alpha}{a^{1/\alpha}(t)} \ and \ B = \frac{\alpha+1}{a^{1/\alpha}(t)},$$

we get,

$$A^{\alpha+1}(t)w(t) \leq A^{\alpha+1}(t_2)w(t_2) - \int_{t_2}^{t} A^{\alpha+1}(s)P_l(s)ds + A(t) - A(t_2). \quad (2.21)$$

It follows from

$$A^{\alpha}(t)w(t) \leq \frac{1}{A(t)}A^{\alpha+1}(t_2)w(t_2) - \frac{1}{A(t)}\int_{t_2}^{t} A^{\alpha+1}(s)P_l(s)ds + 1 \quad (2.22)$$

$$-\frac{A(t_2)}{A(t)}.$$

Taking limit supreme on both sides as$t \to \infty$we obtain$R \leq -Q + 1$.Combining this with the inequality (2.16) we get

$$P + Q \leq 1. \quad (2.23)$$

This is a contradiction. If $z(t)$ satisfies condition (2.2) then by Lemma 2.1 of case (II) we have $\lim_{t \to \infty} x(t) = 0$.This completes the proof.
From Theorem 2.2 we have the following corollary.

Corollary 2.2. Assume that (2.2) holds and $a'(t) \leq 0$ for all $t \geq t_0$.
Let $x(t)$ be a solution of equation (1.1). If

$$Q = \lim_{t\to\infty} \sup \frac{1}{A(t)} \int_{t_0}^{t} A^{\alpha+1}(s) P_l(s) ds > 1, \tag{2.24}$$

then every solution of equation (1.1) is either oscillatory or$\lim_{t\to\infty} x(t) = 0$.

3. Examples

In this section we present some examples to illustrate the main results.

Example 3.1. Consider the differential equation

$$\left(\frac{1}{t^3}\left(\left(x(t) + \frac{1}{3}x\left(t/2\right)\right)''\right)^3\right)' + \frac{1500}{27t^{10}} \max_{[t/2,\, t]} x^3(s) = 0, t \geq 1. \tag{3.1}$$

One can easily verify that all conditions of Theorem 2.1 are satisfied and hence every solution of equation (1.1) isoscillatory.Infact$x(t) = \frac{1}{t}$is one such solution of equation (3.1).

Example 3.2. Consider the differential equation

$$\left(\frac{1}{t}\left(\left(x(t) + \frac{1}{3}x\left(t/3\right)\right)''\right)\right)' + \frac{8}{3t^2} \max_{\left[\frac{t}{3},\, t\right]} x(s) = 0, t \geq 1. \tag{3.2}$$

One can easily verify that all conditions of Theorem 2.2 are satisfiedand hence every solution of equation (1.1) is oscillatory. Infact$x(t) = \frac{1}{t}$is one such solution of equation (3.2).

References

[1] R.Arul and M.Mani,On the oscillation of Third Order quasilinear delay differentialequations with "Maxima" *Malaya Journal of Mat.*(to appear).
[2] D.D.Bainov and S.G.Hristova, Differential Equations with'Maxima', *CRC Press, New* York.2011.
[3] D.Bainov, V.Petrov and V.Proytcheva, Oscillatory andasymptotic behavior of second order neutral differentialequations with 'Maxima', *Dyn. Sys. Appl., 4* (1993), 135-146.

[4] D.Bainov, V.Petrov and V.Proytcheva, Oscillation andnonoscillation of first order neutral differential equations with'Maxima', *SUTJ. Math*., 31 (1995), 17-28.

[5] D.Bainov, V.Petrov and V.Proytcheva, Existence andasymptotic behavior of nonoscillatory solutions of second order neutral differential equations with 'Maxima', *J. Comput.Appl. Math.*, 83 (1997), 237-249.

[6] D.D.Bainov and A.I.Zahariev, Oscillatory and asymptoticproperties of a class of functional differential equations with'Maxima', *Czech. Math*. J.,34 (1984), 247-251

[7] D.Bainov, V.Petrov and V.Proicheva, Oscillation of neutral differential equations with 'Maxima', *Rev. Math*.,8 (1995),171-180.

[8] J.Dzurina,E.Thandapani,S.Tamilvanan,Oscillationof solutions to third order half- linear neutraldifferential equations *Electronic Journal of DifferentialEquations,*Vol.2012(2012),No.29,pp.1-9.ISSN:1072-6691.

[9] Z.Han, T.Li, S.Sun and W.Chen, Oscillation of second orderquasilinear neutral delay differential equations, *J. Appl. Math.Comput*.,40 (2012), 143-152.

[10] T.Li, Z.Han, C.Zhang and S.Sun, On the oscillation of secondorder Emden-Fowler Neutral differential equations, *J. Appl.Math. Comput*.,42(2) (2013), 131-138.

[11] T.Li, S.Sun, Z.Han, B.Han and Y.Sun, Oscillation results forsecond order quasilinear neutral delay differential equations,*Hacettepe J. of Math. and Stat*.,37 (2011), 601-610.

[12] A.R.Magomedev, On some problems of differential equationswith 'Maxima', *Izv. Acad. Sci. Azerb. SSr, Ser. Phys-Techn.and Math. Sci*.,108(1977), 104-108(in Russian)

[13] V.A.Petrov, Nonoscillatory solutions of neutral differentialequations with 'Maxima', *Commun. Appl. Anal.*,2 (1998), 129-142.

[14] E.P.Popov, Automatic Regulation and Control, *Nauka,Moscow.*, 1996.

[15] E.Thandapani and V.Ganesan, Oscillatory and asymptoticbehavior of solution of second order neutral delay differentialequations with “Maxima", *Inter. J. of Pure and Appl.Math*.,78(7) (2012), 1029-1039.