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On Quasi-bilinear Generating Functions of Biorthogonal Polynomials Suggested by Laguerre Polynomials

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Abstract

In this note, we have obtained a novel extension of a bilateral generating functions involving modified biorthogonal polynomials, $Y_n^{\alpha+n}(x;k)$from the existence of quasi-bilinear generating function by group theoretic method. As particular cases, we obtain the corresponding results on generalised Laguerre polynomials.

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1. Introduction

In [1], Carlitz defined Konhauser biorthogonal polynomials $Y_n^\alpha(x;k)$ [2] suggested by Laguerre polynomials [4] as follows:

$$Y_n^\alpha(x;k) = \frac{1}{n!}\sum_{i=0}^{n}\frac{x^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{j+\alpha+1}{k}\right)_n, \tag{1.1}$$

Where $(a)_n$is the pochhammer symbol [5], $\alpha > -1, k$, is a non-zero positive integer.

In [3], the quasi bilateral /(bilinear) generating function is defined by

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n\, p_n^{(\alpha)}(x)\, q_m^{(n)}(u) w^n, \tag{1.2}$$

where a_n, the coefficients are quite arbitrary and $p_n^{(\alpha)}(x), q_m^{(n)}(u)$ are two special functions of orders n, m and of parameters α and n respectively. If $q_m^{(n)}(u) \equiv p_m^{(n)}(u)$, the generating relation is known as quasi bilinear.

The aim at presenting this note is to prove the existence of a more general generating relation from the existence of a quasi-bilinear generating function involving modified biorthogonal polynomials, $Y_n^{\alpha+n}(x;k)$. In [6], Samanta and Chongdar have proved the following theorem on bilateral generating functions involving $Y_n^{\alpha+n}(x;k)$ by group-theoretic method.

Theorem 1 If there exists a unilateral generating relation of the form

$$G(x,w) = \sum_{n=0}^{\infty} a_n Y_n^{\alpha+\mathrm{k}}(\mathrm{x};\mathrm{k})\, w^n \tag{1.3}$$

then

$$(1-kw)^{-\frac{(1+\alpha)}{k}} \exp\left[x\left\{1-(1-kw)^{-\frac{1}{k}}\right\}\right] G\left(x(1-kw)^{-\frac{1}{k}}, wz(1-kw)^{-\frac{k+1}{k}}\right)$$

$$= \sum_{n=0}^{\infty} w^n\, \sigma_n(x,z), \tag{1.4}$$

where

$$\sigma_n(x,z) = \sum_{p=0}^{n} a_p \binom{n}{p} k^{n-p} Y_n^{\alpha+p}(\mathrm{x};\mathrm{k})\, z^p.$$

The importance of the above theorem lies in the fact that whenever one knows a generating relation of the form (1.3) then the corresponding bilateral generating relation can at once be written down from (1.4). So one can get a large number of bilateral generating relations by attributing different suitable values to a_n in (1.3).

In the present paper, we have obtained the following extension of the Theorem1, stated above from the existence of quasi bilinear generating relation.

Theorem 2 If there exists a quasi-bilinear generating relation of the following form

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n Y_n^{\alpha+n}(x;k)\, Y_m^{n}(u;k) w^n \tag{1.5}$$

Then

$$(1-kw)^{-\frac{(1+\alpha)}{k}} \exp\left[x\left\{1-(1-kw)^{-\frac{1}{k}}\right\} - w\right] G\left(x(1-kw)^{-\frac{1}{k}}, u+w, wz(1-kw)^{-\frac{k+1}{k}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!} (-1)^q (n+1)_p\, k^p\, Y_{n+p}^{\alpha+n}(x;k) Y_m^{n+q}(u;k)\, z^n. \tag{1.6}$$

2. Proof of the Theorem

For the biorthogonal polynomials, we consider the following operators:

$$R_1 = xy^{-1}z\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + (k+1)y^{-1}z^2\frac{\partial}{\partial z} + (1-x)y^{-1}z,$$

$$R_2 = v\frac{\partial}{\partial u} - v$$

such that

$$R_1(Y_n^{\alpha+n}(\mathrm{x};\mathrm{k})y^{\alpha}z^n) = k(n+1)Y_{n+1}^{\alpha+n}(\mathrm{x};\mathrm{k})y^{\alpha-1}z^{n+1}, \tag{2.1}$$

$$R_2(Y_m^n(\mathrm{u};\mathrm{k})v^n) = -Y_m^{n+1}(\mathrm{x};\mathrm{k})v^{n+1} \tag{2.2}$$

and

$$e^{wR_1} f(x,y,z) = (1-kwy^{-1}z)^{-\frac{1}{k}}\exp\left[x\left\{1-(1-kwy^{-1}z)^{-\frac{1}{k}}\right\}\right]$$

$$\times f\left(x(1-kwy^{-1}z)^{-\frac{1}{k}}, y(1-kwy^{-1}z)^{-\frac{1}{k}}, z(1-kwy^{-1}z)^{-\frac{k+1}{k}}\right) \tag{2.3}$$

$$e^{wR_2} f(u,v) = \exp(-wv)f(u+vw,v). \tag{2.4}$$

Let us now consider the quasi-bilinear generating function of the following form:

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n Y_n^{\alpha+n}(x;k)Y_m^n(u;k)w^n. \tag{2.5}$$

Replacing w by wvz and multiplying both sides of (2.5) by $y^{\alpha}$and then operating $e^{wR_1}\ e^{wR_2}$ on both sides, we get

$$e^{wR_1}\ e^{wR_2}[y^{\alpha}\ G(x,u,wvz)] = e^{wR_1}\ e^{wR_2}[\textstyle\sum_{n=0}^{\infty} a_n Y_n^{\alpha+n}(x;k)Y_m^n(u;k)y^{\alpha}(wvz)^n]. \tag{2.6}$$

Now the left number of (2.6), with the help of (2.3)and(2.4),becomes

$$(1-kwy^{-1}z)^{-\frac{1+x}{k}}\exp\left[x\left\{1-(1-kwy^{-1}z)^{-\frac{1}{k}}\right\}-vw\right]y^{\alpha}$$

$$\times G\left(x(1-kwy^{-1}z)^{-\frac{1}{k}}, u+vw, wvz(1-kwy^{-1}z)^{-\frac{k+1}{k}}\right). \tag{2.7}$$

The right number of (2.6), with the help of(2.1)and (2.2), becomes

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!}(-1)^q(n+1)_p k^p Y_{n+p}^{\alpha+n}(x;k) Y_m^{n+q}(u;k) y^{\alpha-p} z^{n+p} v^{n+q}. \quad (2.8)$$

Now equating (2.7) and (2.8) and then substituting $\frac{y}{z} = 1, v = 1$,we get

$$(1-kw)^{-\frac{1+\alpha}{k}} \exp\left[x\left\{1-(1-kw)^{-\frac{1}{k}}\right\} - w\right] G\left(x(1-kw)^{-\frac{1}{k}}, u + w, wz(1-kw)^{-\frac{k+1}{k}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!}(-1)^q(n+1)_p k^p Y_{n+p}^{\alpha+n}(x;k) Y_m^{n+q}(u;k) z^n. \quad (2.9)$$

This completes the proof of Theorem 2.

Corollary 1 If we put $m = 0$, we notice that $G(x,u,w)$ becomes $G(x,w)$ since $Y_0^{n+q}(u;k) = 1$. Hence from (2.9), we get

$$(1-kw)^{-\frac{1+x}{k}} \exp\left[x\left\{1-(1-kw)^{-\frac{1}{k}} - w\right\}\right] G\left(x(1-kw)^{-\frac{1}{k}}, wz(1-kw)^{-\frac{k+1}{k}}\right)$$

$$= \exp(-w)\sum_{n=0}^{\infty}\sum_{p=0}^{n} a_{n-p}\binom{n}{p} k^p\, Y_n^{\alpha+n-p}(x;k) w^n z^{n-p}.$$

Therefore we have

$$(1-kw)^{-\frac{1+\alpha}{k}} \exp\left[x\left\{1-(1-kw)^{-\frac{1}{k}}\right\}\right] G\left(x(1-kw)^{-\frac{1}{k}}, wz(1-kw)^{-\frac{k+1}{k}}\right) = \sum_{n=0}^{\infty} w^n \sigma_n(x,z), \quad (2.10)$$

where

$$\sigma_n(x,z) = \sum_{p=0}^{n} a_p \binom{n}{p} k^{n-p}\, Y_n^{\alpha+p}(x;k)\, z^p,$$

which is Theorem 1.

3. Particular Cases

We now proceed to find some particular cases of our Theorem 2.

Case 1 If we put $k = 1$, then $Y_n^{\alpha}(x;k)$ reduces to generalised Laguerre polynomials,$L_n^{\alpha}(x)$. Thus putting $k = 1$in our Theorem 2, we get the following theorem on quasi-bilinear generating relation involving modified Laguerre polynomials.

Theorem 3 If there exists a quasi-bilinear generating relation of the following form

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n\, L_n^{(\alpha+n)}(x) L_m^{(n)}(u)\, w^n \tag{3.1}$$

then

$$(1-w)^{-(1+x)} \exp\left[\frac{-wx}{1-w} - w\right] G\left(\frac{x}{(1-w)},\ u+w, \frac{wz}{(1-w)^2}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n\, \frac{w^{n+p+q}}{p!q!} (-1)^q (n+1)_p\, L_{n+p}^{(\alpha+n)}(x) L_m^{(n+q)}(u)\, z^n. \tag{3.2}$$

Case 2 Putting m = 0 in Theorem 3 and then simplifying, we get the following theorem on bilateral generating relations(found derived in [6]) involving Laguerre polynomials.

Theorem 4 If there exists a unilateral generating relation of the form

$$G(x,w) = \sum_{n=0}^{\infty} a_n\, L_n^{(a+n)}(x)\, w^n \tag{3.3}$$

then

$$(1-w)^{-(1+\alpha)} \exp\left(\frac{-wx}{1-w}\right) G\left(\frac{x}{(1-w)}, \frac{wz}{(1-w)^2}\right) = \sum_{n=0}^{\infty} w^n\, \sigma_n(x,z), \tag{3.4}$$

where

$$\sigma_n(x,z) = \sum_{k=0}^{n} a_k \binom{n}{k} L_n^{(a+k)}(\mathrm{x})\, z^k.$$

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