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Oscillation Solutions to Third Order Half-linear Neutral Difference Equations with "Maxima"

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Abstract

In this paper, we study the oscillation solutions to third order half-linear neutral difference equation with "maxima" of the from

$$\Delta\left(a_n\left(\Delta^2\left(x_n + p_n x_{n-\sigma}\right)\right)^\alpha\right) + q_n \max_{[n-\tau,n]} x_s^\alpha = 0, \qquad (0.1)$$

where α is a ratio of odd positive integers and $\sum_{s=n_0}^{\infty} \frac{1}{a_s^{1/\alpha}} = \infty$. Examples are provided to illustrate the main results.

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1. Introduction

In this paper, we study the oscillation solutions to third order half-linear neutral difference equation with "maxima" of the from

$$\Delta\left(a_n\left(\Delta^2\left(x_n + p_n x_{n-\sigma}\right)\right)^\alpha\right) + q_n \max_{[n-\tau,n]} x_s^\alpha = 0, \quad n \in \mathbb{N}_0 \qquad (1.1)$$

where $\mathrm{N}_0 = \{n_0, n_0 + 1, n_0 + 2, ...\}$ and n_0 is a nonnegative integer subject to the following conditions:

(C_1) α is a ratio of odd positive integers;

(C_2) $\{a_n\}$ is a positive real sequence with $A_n = \sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\alpha}} = \infty$.

(C_3) $\{q_n\}$ and $\{p_n\}$ are positive real sequences with $0 \le p_n \le p < 1$;

(C_4) τ and σ are positive integers.

Let $\theta = \max\{\tau, \sigma\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ satisfying equation (1.1) for all $n \ge n_0 - \theta$. Such a solution is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

In [1, 2, 7], the authors investigated the oscillatory properties of solutions of third order neutral difference equations. But very few results available in the literature dealing with the oscillatory and asymptotic behavior of solutions of neutral difference equations with “maxima”, see [3, 5, 6], and the references cited therein. Therefore, in this paper, we present some sufficient conditions for the oscillation of all solutions of equation (1.1). The result established in this paper are discrete analogue of that in [4].

In Section 2, we present some sufficient conditions for the oscillation of all solutions of equation (1.1). In Section 3, we provide some examples to illustrate the main results.

2. Oscillation Results

In this section, we obtain a oscillation criteria for equation (1.1). We begin with some useful lemmas, which some useful lemmas, which we intend to use later. For each solution $\{x_n\}$ of equation (1.1), we define the corresponding sequence $\{z_n\}$ by

$$z_n = x_n + p_n x_{n-\sigma}. \tag{2.1}$$

The following Lemmas extracted from [7].

Lemma 2.1. Let $\{x_n\}$ be a positive solution of equation (1.1), then there are only the following two cases for $\{z_n\}$ defined in (2.1):

(i) $z_n > 0,\ \Delta z_n > 0,\ \Delta^2 z_n > 0$ and $\Delta\left(a_n \Delta^2 z_n\right) \le 0;$

(ii) $z_n > 0,\ \Delta z_n < 0,\ \Delta^2 z_n > 0$ and $\Delta\left(a_n \Delta^2 z_n\right) \le 0$

for $n \ge n_1 \in \mathbb{N}_0$, where n_1 is sufficiently large.

Lemma 2.2. Let $\{x_n\}$ be a positive solution of equation (1.1), and let the corresponding function $\{z_n\}$ satisfies the Case (II) of Lemma 2.1. If

$$\sum_{n=n_0}^{\infty} \sum_{s=n}^{\infty} \left[\frac{1}{a_s} \sum_{t=s}^{\infty} q_t \right]^{1/\alpha} = \infty, \tag{2.2}$$

then $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = 0.$

Lemma 2.3. *Assume that* $u_n > 0,\ \Delta u_n \geq 0$ *and* $\Delta\left(a_n\left(\Delta u_n\right)^{\alpha}\right) \leq 0$ for all $n \geq n_0$. Then for each $\ell \in (0,1)$ there exists a $N \in \mathbb{N}_0$ such that

$$\frac{u_{n-\tau}}{A_{n-\tau}} \geq \ell \frac{u_n}{A_n} \text{ for all } n \geq N.$$

Lemma 2.4. *Assume that* $z_n > 0,\ \Delta z_n \geq 0,\ \Delta^2 z_n > 0,$ *and* $\Delta\left(a_n\left(\Delta^2 z_n\right)^{\alpha}\right) \leq 0$ for all $n \geq N$. Then

$$\frac{z_{n+1}}{\Delta z_n} \geq \frac{a_n^{1/\alpha} A_n}{2} \text{ for all } n \geq N.$$

Lemma 2.5. *Assume that* $\Delta z_n \geq 0,\ \Delta^2 z_n > 0,$ *and* $\Delta\left(a_n\left(\Delta^2 z_n\right)^{\alpha}\right) \leq 0$ for all $N \in \mathbb{N}_0$. Then

$$a_n^{1/\alpha} A_n \frac{\Delta^2 z_n}{\Delta z_n} \leq 1 \text{ for all } n \geq N.$$

Lemma 2.6. If $\lim\limits_{n\to\infty} \dfrac{a_n^{-1/\alpha}}{A_n} = 0$, then

$$\lim_{n\to\infty} \frac{1}{A_{n+1}} \sum_{s=N}^{\infty} \left(1 + \frac{a_s^{-1/\alpha}}{A_s}\right)^{\alpha(\alpha+1)} \frac{1}{a_{s+1}^{1/\alpha}} = 1.$$

Lemma 2.7. The sequence $\{x_n\}$ is an eventually negative solution of equation (1.1) if and only if $\{-x_n\}$ is an eventually positive solution of the equation

$$\Delta\left(a_n\left(\Delta^2\left(x_n + p_n x_{n-\sigma}\right)\right)^{\alpha}\right) + q_n \max_{[n-\tau,n]} x_s^{\alpha} = 0.$$

The assertion of Lemma 2.7 can be verified easily.

Now, we present the main results. For simplicity we introduce the following notations:

$$P = \liminf_{n\to\infty} A_{n+1}^{\alpha} \sum_{s=n}^{\infty} P_\ell(s),$$

$$Q = \limsup_{n\to\infty} \frac{1}{A_{n+1}} \sum_{s=n_0}^{\infty} A_s^{\alpha+1} P_\ell(s)$$

where

$$P_\ell(s) = q_n \ell^{\alpha} \frac{A_{n-\tau}^{2\alpha}}{A_n^{\alpha} 2^{\alpha}} a_{n-\tau} \max_{[n-\tau,n]} \left(1 - p_s\right)^{\alpha}$$

with $\ell \in (0,1)$. Moreover, for $\{z_n\}$ satisfying the Case (I) of Lemma 2.1, we define

$$w_n = a_n\left(\frac{\Delta^2 z_n}{z_n}\right)^{\alpha}, \quad n \geq N,$$

$$r = \liminf_{n\to\infty} A_{n+1}^{\alpha} w_{n+1},$$

and

$$R = \limsup_{n\to\infty} A_{n+1}^{\alpha} w_{n+1}.$$

Theorem 2.1. Assume that condition (2.2) holds. If

$$P > \frac{\alpha}{(\alpha+1)^{(\alpha+1)}}, \tag{2.3}$$

then every solution $\{x_n\}$ of equation (2.1) is either oscillatory or $\lim_{n\to\infty} x_n = 0$.

Proof: Assume that $\{x_n\}$ is a nonoscillatory solution of equation (1.1). We can assume without loss of generality, that $\{x_n\}$ is positive and corresponding $\{z_n\}$ satisfies Case (I) of Lemma 2.1. First note that

$$x_n = z_n - p_n x_{n-\sigma} \geq (1 - p_n) z_n$$

or

$$\max_{[n-\tau,n]} x_s^{\alpha} \geq z_n^{\alpha} \max_{[n-\tau,n]} (1-p_s)^{\alpha} \geq z_{n+1-\tau}^{\alpha} \max_{[n-\tau,n]} (1-p_s)^{\alpha}.$$

Since $\{z_n\}$ is increasing and $\tau \geq 1$. Using the above inequality in equation (1.1), we obtain

$$\Delta\left(a_n\left(\Delta^2 z_n\right)^{\alpha}\right) \leq -q_n z_{n+1-\tau}^{\alpha} \max_{[n-\tau,n]} (1-p_s)^{\alpha} \leq 0. \tag{2.4}$$

From the definition of $\{w_n\}$ we see that $w_n > 0$ and from (1.1) we have

$$w_n = \frac{\Delta\left(a_n\left(\Delta^2 z_n\right)^{\alpha}\right)}{(\Delta z_n)^{\alpha}} - \frac{a_{n+1}\left(\Delta^2 z_{n+1}\right)^{\alpha}}{(\Delta z_{n+1})^{\alpha}(\Delta z_n)^{\alpha}} \Delta(\Delta z_n)^{\alpha}$$

$$\leq -q_n z_{n+1-\tau}^{\alpha} \max_{[n-\tau,n]} (1-p_s)^{\alpha} \frac{1}{(\Delta z_n)^{\alpha}} - \frac{\alpha}{a_{n+1}^{1/\alpha}} w_{n+1}^{1+1/\alpha}. \tag{2.5}$$

By Lemma 2.3 with $u_n = \Delta z_n$, we have

$$\frac{1}{\Delta z_n} \geq \ell \frac{A_{n-\tau}}{A_n} \frac{1}{\Delta z_{n-\tau}}, \quad n \geq N,$$

which with (2.5) gives

$$\Delta w_n \le -q_n \ell^\alpha \left(\frac{A_{n-\tau}}{A_n}\right)^\alpha \left(\frac{1}{\Delta z_{n-\tau}}\right)^\alpha z_{n+1-\tau}^\alpha \max_{[n-\tau,n]} \left(1-p_s\right)^\alpha \frac{1}{\left(\Delta z_n\right)^\alpha} - \frac{\alpha}{a_{n+1}^{1/\alpha}} w_{n+1}^{1+1/\alpha}.$$

Using the fact from Lemma 2.4 that $\frac{z_{n+1}}{\Delta z_n} \ge \frac{a_n^{1/\alpha} A_n}{2}$, we have

$$\Delta w_n \le -q_n \ell^\alpha \frac{A_{n-\tau}^{2\alpha}}{A_n^\alpha 2^\alpha} a_{n-\tau} \max_{[n-\tau,n]} \left(1-p_s\right)^\alpha - \frac{\alpha}{a_{n+1}^{1/\alpha}} w_{n+1}^{1+1/\alpha}$$

or

$$\Delta w_n + P_\ell\left(n\right) + \frac{\alpha}{a_{n+1}^{1/\alpha}} w_{n+1}^{1+1/\alpha} \le 0 \text{ for all } n \ge N. \tag{2.6}$$

Since $P_\ell\left(n\right) > 0$ and $w_n > 0$ for $n \ge N$, w have from (2.6) that $\Delta w_n \le 0$ and

$$-\frac{\Delta w_n}{\alpha w_{n+1}^{1+1/\alpha}} \ge \frac{1}{a_{n+1}^{1/\alpha}} \text{ for all } n \ge N. \tag{2.7}$$

Summing the last inequality from N to $n-1$, and using the decreasing property of w_n, we obtain

$$\frac{-w_n + w_N}{\alpha w_{n+1}^{1+1/\alpha}} \ge \sum_{s=N}^{n-1} \frac{1}{a_{s+1}^{1/\alpha}}$$

or

$$w_n \le \left(\frac{w_N}{\alpha \sum_{s=N}^{n-1} \frac{1}{a_{s+1}^{1/\alpha}}} \right)^{\frac{\alpha}{\alpha+1}},$$

which inview of $\left(C_2\right)$ implies that $\lim_{n\to\infty} w_n = 0$. On the otherhand, from the definition of w_n, and Lemma 2.5, we see that

$$A_n^\alpha w_n = a_n \left(\frac{A_n \Delta^2 A_n}{\Delta z_n}\right)^\alpha \le 1^\alpha.$$

Then

$$0 \le r \le R < 1. \tag{2.8}$$

Let $\varepsilon > 0$, then from the definition of P and r, we can choose an integer $n_2 \geq N$ sufficiently large that

$$A_{n+1}^{\alpha}\sum_{s=n}^{\infty}P_{\ell}(s) \geq P-\varepsilon \text{ and } A_{n+1}^{\alpha}w_{n+1} \geq r-\varepsilon \text{ for } n \geq n_3. \tag{2.9}$$

Summing (2.6) from n to ∞, and using the fact that $\lim_{n\to\infty} w_n = 0$, we have

$$w_{\infty} - w_{n+1} + \sum_{s=n}^{\infty}P_{\ell}(s) + \alpha\sum_{s=n}^{\infty}\frac{w_{s+1}^{1+1/\alpha}}{a_{s+1}^{1/\alpha}} \leq 0$$

or

$$w_{n+1} \geq \sum_{s=n}^{\infty}P_{\ell}(s) + \alpha\sum_{s=n}^{\infty}\frac{w_{s+1}^{1+1/\alpha}}{a_{s+1}^{1/\alpha}}, \quad n \geq n_2. \tag{2.10}$$

Multiplying the last inequality by A_{n+1}^{α}, we have

$$A_{n+1}^{\alpha}w_{n+1} \geq A_{n+1}^{\alpha}\sum_{s=n}^{\infty}P_{\ell}(s) + \alpha A_{n+1}^{\alpha}\sum_{s=n}^{\infty}\frac{w_{s+1}^{1+1/\alpha}}{a_{s+1}^{1/\alpha}}$$

$$\geq P-\varepsilon + (r-\varepsilon)^{1+1/\alpha}\,\alpha A_{n+1}^{\alpha}\sum_{s=n}^{\infty}\frac{\Delta A_{s+1}}{A_{s+1}^{1/\alpha}}. \tag{2.11}$$

From (2.11), and $\sum_{s=n}^{\infty}\alpha\frac{\Delta A_{s+1}}{A_{s+1}^{1/\alpha}} \geq \alpha\int_{A_{n+1}}^{\infty}\frac{ds}{s^{\alpha}} \geq \frac{1}{A_{n+1}^{\alpha}}$, we have

$$A_{n+1}^{\alpha}w_{n+1} \geq P-\varepsilon + (r-\varepsilon)^{1+1/\alpha}.$$

Taking limit infimum on both sides, we obtain that

$$r \geq P-\varepsilon + (r-\varepsilon)^{1+1/\alpha}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain that desired that

$$P \leq r - r^{1+1/\alpha}. \tag{2.12}$$

Using the inequality $Bu - Cu^{\frac{1+\alpha}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{C^{\alpha}}$, with $B = C = 1$ and $u = r$, we obtain that

$$P \leq \frac{\alpha}{(\alpha+1)^{(\alpha+1)}}$$

which contradicts (2.3).

If $\{z_n\}$ satisfies Case (II) of Lemma 2.1, then by the condition (2.2) we have $\lim_{n\to\infty} x_n = 0$. This completes the proof.

From Theorem 2.1 we have the following corollary.

Corollary 2.1. Assume that (2.2) holds. If

$$\liminf_{n\to\infty} A_{n+1}^{\alpha}\sum_{s=n_0}^{\infty} q_s a_{s-\tau}\frac{A_{s-\tau}^{2\alpha}}{A_s^{\alpha}}\max_{[s-\tau,s]}(1-p_t)^{\alpha} > \left(\frac{2}{\ell}\right)^{\alpha}\frac{\alpha}{(\alpha+1)^{(\alpha+1)}},$$

then every solution $\{x_n\}$ of equation (2.1) is either oscillatory or $\lim_{n\to\infty} x_n = 0.$

Theorem 2.2. Assume that condition (2.2) holds. If

$$P + Q > 1, \qquad (2.13)$$

then every solution $\{x_n\}$ of equation (2.1) is either oscillatory or $\lim_{n\to\infty} x_n = 0.$

Proof: Assume that $\{x_n\}$ is a nonoscillatory solution of equation (1.1). We can assume without loss of generality, that $\{x_n\}$ is positive solution of equation (1.1). Let the corresponding $\{z_n\}$ satisfies Case (I) of Lemma 2.1. Proceeding as in the proof of Theorem 2.1, we obtain (2.6). Now multiplying (2.6) by A_{n+1}^{α} and summing from n_2 to n, and then using summation by parts formula, we obtain

$$A_{n+1}^{\alpha+1}w_{n+1} \le A_{n_2}^{\alpha+1}w_{n_2} - \sum_{s=n_2}^{n} A_s^{\alpha+1}P_\ell(s) + \sum_{s=n_2}^{n} w_{s+1}\Delta A_s^{\alpha+1} - \sum_{s=n_2}^{n}\alpha A_s^{\alpha+1}\frac{w_{s+1}^{\frac{\alpha+1}{\alpha}}}{a_{s+1}^{1/\alpha}}$$

$$\le A_{n_2}^{\alpha+1}w_{n_2} - \sum_{s=n_2}^{n} A_s^{\alpha+1}P_\ell(s) + \sum_{s=n_2}^{n}(\alpha+1)A_{s+1}^{\alpha}\Delta A_s w_{s+1}^{\frac{\alpha+1}{\alpha}} - \sum_{s=n_2}^{n}\alpha A_s^{\alpha+1}\Delta A_s w_{s+1}^{\frac{\alpha+1}{\alpha}}.$$

Using the inequality $Bu - Cu^{\frac{1+\alpha}{\alpha}} \le \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{C^{\alpha}}$, with $u = w_{n+1}$, $C = \alpha A_s^{\alpha+1}\Delta A_s$ and $B = (\alpha+1)A_{s+1}^{\alpha}\Delta A_s$, we obtain that

$$A_{n+1}^{\alpha+1}w_{n+1} \le A_{n_2}^{\alpha+1}w_{n_2} - \sum_{s=n_2}^{n} A_s^{\alpha+1}P_\ell(s) + \sum_{s=n_2}^{n}\left(\frac{A_{s+1}}{A_s}\right)^{\alpha(\alpha+1)}\Delta A_s$$

$$\le A_{n_2}^{\alpha+1}w_{n_2} - \sum_{s=n_2}^{n} A_s^{\alpha+1}P_\ell(s) + \sum_{s=n_2}^{n}\left(\frac{A_{s+1}}{A_s}\right)^{\alpha(\alpha+1)}\frac{1}{a_{s+1}^{1/\alpha}}.$$

It follows that

$$A_{n+1}^{\alpha+1}w_{n+1} \le \frac{A_{n_2}^{\alpha+1}w_{n_2}}{A_{n+1}} - \frac{1}{A_{n+1}}\sum_{s=n_2}^{n} A_s^{\alpha+1}P_\ell(s) + \frac{1}{A_{n+1}}\sum_{s=n_2}^{n}\left(1+\frac{a_s^{-1/\alpha}}{A_s}\right)^{\alpha(\alpha+1)}\frac{1}{a_{s+1}^{1/\alpha}}.$$

Taking limit supremum on both sides and using Lemma 2.6, we obtain

$$R \le -Q+1.$$

Combining this with the inequality in (2.12) and (2.8) we obtain

$$P \le r - r^{\frac{\alpha+1}{\alpha}} \le r \le R \le -Q+1$$

or

$$P+Q \le 1,$$

which contradicts (2.13)

If $\{z_n\}$ satisfies Case (II) of Lemma 2.1, then by the condition (2.2) we have $\lim_{n\to\infty} x_n = 0$. This completes the proof.

From Theorem 2.2 we have the following corollary.

Corollary 2.2. Assume that (2.2) holds and $\lim_{n\to\infty} \frac{a_n^{-1/\alpha}}{A_n} = 0$. If

$$Q = \lim_{n\to\infty}\sup \frac{1}{A_{n+1}} \sum_{s=n_0}^{\infty} A_s^{\alpha+1} P_\ell(s) > 1,$$

then every solution $\{x_n\}$ of equation (2.1) is either oscillatory or $\lim_{n\to\infty} x_n = 0$.

3. 3. Examples

In this section we present some examples to illustrate the main results.

Example 3.1. Consider the third order difference equation

$$\Delta\left(n(n+1)\Delta^2\left(x_n + \frac{1}{2}x_{n-1}\right)\right) + n\max_{[n-1,n]} x_s = 0, \; n \ge 1. \tag{3.1}$$

Here $\alpha = 1$, $a_n = n(n+1)$, $p_n = \frac{1}{2}, q_n = n,$, and $\tau = \sigma = 1$. It is easy to see that all conditions of Corollary 2.1 are satisfied and hence every solution of equation (3.1) is either oscillatory or converge to zero as $n \to \infty$.

Example 3.2. Consider the third order difference equations

$$\Delta\left(n\left(\Delta^2\left(x_n + \frac{1}{3}x_{n-1}\right)\right)^3\right) + \frac{10}{n^6}\max_{[n-1,n]} x_s^3 = 0, \; n \ge 1. \tag{3.2}$$

Here $\alpha = 3$, $a_n = n$, $p_n = \frac{1}{3}, q_n = \frac{10}{n^6}$, and $\tau = \sigma = 1$. It is easy to see that all conditions of Corollary 2.1 are satisfied and hence every solution of equation (3.2) is either oscillatory or converge to zero as $n \to \infty$.

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