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Some Asymptotic Properties of Third Order Nonlinear Difference Equations

R.Arul[1] and R. Kodeeswaran[2]

Department of Mathematics Kandaswami Kandar's College
Velur – 638 182, Namakkal Dt. Tamil Nadu, India.

Abstract

We present new criteria for asymptotic properties of third order nonlinear difference equation of the form

$$\Delta^2(a_n(\Delta x_n)^\alpha) + p_n x_{\tau(n)}^\beta = 0$$

where $\{a_n\}$ and $\{p_n\}$ are non negative real sequences and $\{\tau(n)\}$ is a sequence of positive integers, α and β are ratios of odd positive integers. Our results are based on the suitable Comparison theorem, in which we deduce properties of the third order difference equation from the second order difference inequality. Some examples are provided to illustrate the main results.

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1 Introduction

In this paper, we are concerned with oscillatory and asymptotic behavior of solutions of the third order nonlinear difference equation of the form

$$\Delta^2(a_n(\Delta x_n)^\alpha) + p_n x_{\tau(n)}^\beta = 0, \qquad n \in N_0 \tag{1.1}$$

where $N_0 = \{n_0, n_0+1, n_0+2, \ldots\}$, n_0 is a non negative integer and $\{a_n\}$ and $\{p_n\}$ are non negative real sequences and $\{\tau(n)\}$ is a sequence of positive integers, α and β are ratios of odd positive integers.

Throughout the paper it is assumed that

$$R_n = \sum_{s=N_0}^{n-1} a_s^{-1/\alpha} \to \infty \quad \text{as} \quad n \to \infty . \tag{1.2}$$

By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ which is defined $n \in N_0$ and satisfying equation (1.1) for all $n \in N_0$. A nontrivial solution $\{x_n\}$ of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

The oscillatory behavior of solution of third order delay difference equations have been investigated by several authors, see for examples [3,4,5,8,14,15] and references quoted therein. Following this trend, in this paper, we offer new technique for investigation of asymptotic properties of solutions of equation (1.1) from certain positive solution of second order difference inequality.

In Section 2, we establish some results on the nonoscillatory properties of equation (1.1), and in Section 3 we provide some examples to illustrate the main results.

2 Main Results

We begin with the following lemma.

Lemma 2.1. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Then $\{x_n\}$ satisfies eventually, one of the following conditions;

(I) $x_n \Delta x_n < 0$, $x_n \Delta(a_n(\Delta x_n)^\alpha) > 0$, $x_n \Delta^2(a_n(\Delta x_n)^\alpha) < 0$;

(II) $x_n \Delta x_n > 0$, $x_n \Delta(a_n(\Delta x_n)^\alpha) > 0$, $x_n \Delta^2(a_n(\Delta x_n)^\alpha) < 0$.

The proof of the lemma can be found in [6].

Definition 2.1. We say that equation (1.1) has property (A) if every nonoscillatory solution of equation (1.1) satisfies case (I) of Lemma 2.1.

Lemma 2.2. Let $m > 0$ and $\tau(n) = n + m$. Assume that $x_n > 0$, $\Delta x_n > 0$, and $\Delta(a_n(\Delta x_n)^\alpha) > 0$ eventually. Then for arbitrary $k \in (0,1)$,

$$x_{n+m} \geq k \frac{R_{n+m}}{R_n} x_n \tag{2.1}$$

eventually.

The proof of the lemma can be found in [6].

Define $p_1(n) = \frac{R_{n+m}^\beta}{R_n^\beta} p_n$.

Theorem 2.1. Let $m > 0$ and $\tau(n) = n + m$. If for some $c \in (0,1)$ the second order difference inequality

$$\Delta\left(\frac{1}{p_1^{1/\beta}(n)} (\Delta z_n)^{1/\beta}\right) + c \frac{n^{1/\alpha}}{a_n^{1/\alpha}} \left(z_n^{1/\alpha}\right) \leq 0 \tag{2.2}$$

has no solution satisfying

$$z_n > 0\,,\quad \Delta z_n < 0 \quad \text{and} \quad \Delta\left(\frac{1}{p_1^{1/\beta}(n)}(\Delta z_n)^{1/\beta}\right) < 0\,, \tag{2.3}$$

then equation (1.1) has property (A).

Proof. Assume that there is a nonoscillatory solution $\{x_n\}$ of equation (1.1) satisfying case (II) of Lemma 2.1. By using (2.1) in equation (1.1), we obtain

$$\Delta^2(a_n(\Delta x_n)^\alpha) + k^\beta p_n \frac{R_{n+m}^\beta}{R_n^\beta} x_n^\beta \le 0\,. \tag{2.4}$$

On the other hand, it follows from the monotonicity of $y_n = \Delta(a_n(\Delta x_n)^\alpha)$ that,

$$a_n(\Delta x_n)^\alpha \ge \sum_{s=N}^{n-1} y_s \ge y_n(n-N) \ge c_1(n)y_n$$

eventually, where $c_1 \in (0,1)$. Then

$$(\Delta x_n)^\alpha \ge \frac{1}{a_n} c_1(n)y_n$$

or

$$\Delta x_n \ge \frac{1}{a_n^{1/\alpha}} c_1^{1/\alpha}\, n^{1/\alpha}\, y_n^{1/\alpha}\,. \tag{2.5}$$

Summing up both sides the inequality (2.5) from N to $n-1$, we have

$$x_n \ge c_1^{1/\alpha} \sum_{s=N}^{n-1} \frac{s^{1/\alpha}}{a_s^{1/\alpha}}\, y_s^{1/\alpha}\,. \tag{2.6}$$

Using (2.6) in (2.4) we obtain

$$\Delta y_n + k^\beta p_1(n) c_1^{\beta/\alpha} \left(\sum_{s=N}^{n-1} \frac{s^{1/\alpha}}{a_s^{1/\alpha}}\, y_s^{1/\alpha}\right)^\beta \le 0.$$

Summing up both sides the last inequality from n to ∞ we have

$$y_n \ge c \sum_{s=n}^{\infty} p_1(s) \left(\sum_{j=N}^{s-1} \frac{j^{1/\alpha}}{a_j^{1/\alpha}} \left(\Delta a_j(\Delta x_j)^\alpha\right)^{1/\alpha}\right)^\beta \tag{2.7}$$

where $c = c_1^{\beta/\alpha} k^\beta$.

Let us denote the right hand side of (2.7) by z_n. Then $y_n \ge z_n > 0$ and z_n satisfies (2.3), and we obtain

$$\Delta\left(\frac{1}{p_1^{1/\beta}(n)}(\Delta z_n)^{1/\beta}\right) + c\frac{n^{1/\alpha}}{a_n^{1/\alpha}}\left(z_n^{1/\alpha}\right) = 0\,. \tag{2.8}$$

Consequently, z_n is a solution of the difference inequality (2.2), which is a contradiction. This completes the proof. ◘

For our next results, define

$$p_2(n) = p(n+m) \tag{2.9}$$

Theorem 2.2. Let $\tau(n) = n - m$. If for some $c \in (0,1)$, the second order difference inequality

$$\Delta\left(\frac{1}{p_2^{1/\beta}(n)}(\Delta z_n)^{1/\beta}\right) + c\frac{n^{1/\alpha}}{a_n^{1/\alpha}}z_n^{1/\alpha} \leq 0 \tag{2.10}$$

has no solution satisfying

$$z_n > 0, \quad \Delta z_n < 0 \quad \text{and} \quad \Delta\left(\frac{1}{p_2^{1/\beta}(n)}(\Delta z_n)^{1/\beta}\right) < 0, \tag{2.11}$$

then equation (1.1) has property (A).

Proof. Let $\{x_n\}$ be a positive solution of equation (1.1) satisfying case (II) of Lemma 2.1. By summing equation (1.1) from n to ∞ we obtain

$$\Delta(a_n(\Delta x_n)^\alpha) \geq \sum_{s=n}^{\infty} p_s\, x^\beta(s-m). \tag{2.12}$$

Denote $s - m = u$, we have

$$\Delta(a_n(\Delta x_n)^\alpha) \geq \sum_{u=n-m}^{\infty} p(u+m)\, x_u^\beta \geq \sum_{s=n}^{\infty} p(s+m)\, x_s^\beta. \tag{2.13}$$

Using (2.5) in (2.13) we have,

$$y_n \geq \sum_{s=n}^{\infty} p(s+m)\left(\sum_{j=N}^{s-1}\frac{1}{a_j^{1/\alpha}}c_1^{1/\alpha}j^{1/\alpha}y_j^{1/\alpha}\right)^\beta$$

or

$$y_n \geq \sum_{s=n}^{\infty} p_s(s)\left(\sum_{j=N}^{s-1}\frac{1}{a_j^{1/\alpha}}c_1^{1/\alpha}j^{1/\alpha}y_j^{1/\alpha}\right)^\beta.$$

Let us denote the right hand side of (2.14) by z_n . Then similarly as in the proof of Theorem 2.1, we can verify that z_n is a positive solution of inequality (2.10) and it satisfies (2.11), which is a contradiction to our assumption. This complete the proof. ◻

Now, we eliminate solution of inequalities (2.2) and (2.10) satisfying (2.3) and (2.11), to obtain sufficient condition for property (A) of equation (1.1). This inequalities (2.2) and (2.10) have the same form, we present one general criteria and adapt them for both inequalities.

Consider the difference equation

$$\Delta(r_n(\Delta z_n)^\alpha) + b_n z_n^\gamma = 0 \tag{2.15}$$

where α and γ are ratios of odd positive integers, and $\{r_n\}$ is a positive real sequence and $\{b_n\}$ for nonnegative real sequence. Define

$$e_n = \sum_{s=n}^{\infty} r_s^{-1/\alpha}.$$

Theorem 2.3. Assume that $\alpha \geq \gamma$. If

$$\sum_{s=N}^{\infty} \frac{1}{r_s^{1/\alpha}} \left(\sum_{j=N}^{s-1} b_j \right)^{1/\alpha} = \infty \tag{2.16}$$

and

$$\lim_{n\to\infty} sup \sum_{s=N}^{n-1} \left(e_s^{\alpha} b_s - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \left(\frac{\alpha}{\gamma}\right)^{\alpha} \frac{1}{r_s^{1/\alpha}} \frac{1}{e_s} \right) > 0, \tag{2.17}$$

then equation (2.15) has no solution satisfying

$$z_n > 0\,, \quad \Delta z_n < 0 \quad \text{and} \quad \Delta(r_n(\Delta z_n)^{\alpha}) < 0\,. \tag{2.18}$$

Proof. From inequality (2.17), we have

$$\sum_{s=N}^{n-1} e_s^{\alpha} b_s > \sum_{s=N}^{n-1} \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \left(\frac{\alpha}{\gamma}\right)^{\alpha} \frac{1}{r_s^{1/\alpha}} \frac{1}{e_s}\,. \tag{2.19}$$

Let $\{z_n\}$ be a positive solution of equation (2.15) satisfying (2.18). Then we claim that $\lim_{n\to\infty} z_n = 0$. If not, then $z_n \geq l > 0$ eventually. From equation (2.15), we obtain

$$-r_n(\Delta z_n)^{\alpha} \geq \sum_{s=N}^{n-1} b_s z_s^{\gamma} \geq l^{\gamma} \sum_{s=N}^{n-1} b_s\,.$$

Evaluating Δz_n and summing up both sides the above inequality from N to $n-1$, we obtain

$$z_N \geq \sum_{s=N}^{n-1} \frac{1}{r_s^{1/\alpha}} l^{\gamma/\alpha} \left(\sum_{j=N}^{s-1} b_j \right)^{1/\alpha}.$$

Letting $n \to \infty$, we get contradiction with (2.16), therefore we conclude that $\lim_{n\to\infty} z_n = 0$. Define

$$w_n = \frac{r_n(\Delta z_n)^{\alpha}}{z_n^{\gamma}}\,. \tag{2.20}$$

Then

$$\Delta w_n = \frac{\Delta(r_n(\Delta z_n)^{\alpha})}{z_n^{\gamma}} - \gamma w_{n+1} t^{\gamma-1} \frac{\Delta z_n}{z_n^{\gamma}}$$

where $z_n < t < z_{n+1}$. Further, we have

$$\Delta w_n \leq -b_n - \gamma w_{n+1} \frac{\Delta z_n}{z_n}$$

$$= -b_n - \gamma w_{n+1} \frac{r_n^{1/\alpha} \Delta z_n}{z_n} \frac{1}{r_n^{1/\alpha}}. \tag{2.21}$$

Clearly, $\lim_{n\to\infty} z_n = 0$ and $\alpha \geq \gamma$, $z_n \leq z_n^{\gamma/\alpha}$. We derive

$$r_{n+1}^{1/\alpha}(\Delta z_{n+1}) \geq r_n^{1/\alpha}(\Delta z_n) \quad \text{and} \quad \frac{1}{r_{n+1}^{\gamma/\alpha}} \geq \frac{1}{z_n}.$$

By using the above inequalities in (2.21) we obtain

$$\Delta w_n \le -b_n - \gamma w_{n+1}^{(1+1/\alpha)} \frac{1}{r_n^{1/\alpha}} . \qquad (2.22)$$

Multiplying (2.22) by e_n^α and summing from N to $n-1$, we obtain

$$e_n^\alpha w_n - e_N^\alpha w_N + \alpha \sum_{s=N}^{n-1} w_s \frac{e_s^{\alpha-1}}{r_s^{1/\alpha}} \le -\sum_{s=N}^{n-1} b_s\, e_s^{\alpha-1} - \gamma \sum_{s=N}^{n-1} w_{s+1}^{(1+1/\alpha)} e_s^\alpha r_s^{1/\alpha}$$

$$\sum_{s=N}^{n-1} b_s\, e_s^\alpha + \alpha \sum_{s=N}^{n-1} w_s \frac{e_s^{\alpha-1}}{r_s^{1/\alpha}} + \gamma \sum_{s=N}^{n-1} w_{s+1}^{(1+1/\alpha)} e_s^\alpha r_s^{1/\alpha} \le e_N^\alpha w_N - e_n^\alpha w_n$$

$$\sum_{s=N}^{n-1} b_s\, e_s^\alpha + \alpha \sum_{s=N}^{n-1} \frac{e_s^{\alpha-1}}{r_s^{1/\alpha}} \left(w_s + w_{s+1}^{(1+1/\alpha)} e_s \right) \le -e_n^\alpha w_n .$$

By using

$$w_s + w_{s+1}^{(1+1/\alpha)} \ge \left(\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \cdot \frac{1}{A^\alpha} \right) \quad \text{and} \quad A = \frac{\gamma}{\alpha} e_s$$

we obtain

$$\sum_{s=N}^{n-1} b_s\, e_s^\alpha \le \sum_{s=N}^{n-1} \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \left(\frac{\alpha}{\gamma} \right)^\alpha \frac{1}{r_s^{1/\alpha}} \frac{1}{e_s} - e_n^\alpha w_n , \qquad (2.23)$$

which is a contradiction with (2.19). This completes the proof.

□

Corollary 2.1. Let (2.16) holds. Assume that $\alpha \ge \gamma$. If

$$\lim_{n\to\infty} inf\; e_n^{\alpha+1} b_n a_n^{1/\alpha} > \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \left(\frac{\alpha}{\gamma} \right)^\alpha , \qquad (2.24)$$

then equation (2.15) has no solution satisfying (2.18).

Proof. It follows from (2.24) that

$$e_n^\alpha b_n > \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \left(\frac{\alpha}{\gamma} \right)^\alpha \frac{1}{e_n a_n^{1/\alpha}}$$

or

$$e_n^\alpha b_n - \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \left(\frac{\alpha}{\gamma} \right)^\alpha \frac{1}{e_n a_n^{1/\alpha}} > 0 . \qquad (2.25)$$

Summing up both sides the inequality (2.25) from N to $n-1$ we obtain

$$\sum_{s=N}^{n-1} \left(e_s^\alpha b_s - \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \left(\frac{\alpha}{\gamma} \right)^\alpha \frac{1}{e_n a_n^{1/\alpha}} \right) > 0 .$$

Taking *limsup* on the both sides,

$$\lim_{n\to\infty} sup \sum_{s=N}^{n-1} \left(e_s^\alpha b_s - \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \left(\frac{\alpha}{\gamma} \right)^\alpha \frac{1}{e_n a_n^{1/\alpha}} \right) > 0 .$$

By Theorem 2.3, we conclude that inequality (2.15) has no solution satisfying (2.18). This completes the proof.

□

Theorem 2.4. Let $\alpha > \beta$ and $\tau(n) = n + m$. If

$$\sum_{s=N}^{\infty} \frac{R_{s+m}^{\beta}}{R_s^{\beta}} p_s \left(\sum_{j=N}^{s-1} \frac{j^{1/\alpha}}{r_j^{1/\alpha}} \right)^{\beta} = \infty, \tag{2.26}$$

and

$$\lim_{n\to\infty} inf \left(\sum_{s=N}^{\infty} \frac{R_{s+m}^{\beta}}{R_s^{\beta}} p_s \right)^{1+1/\beta} \frac{n^{1/\alpha}}{r_n^{1/\alpha}} \frac{R_n^{\beta}}{R_{n+m}^{\beta}} > \left(\frac{1}{\beta+1} \right)^{1+1/\beta} \left(\frac{\alpha}{\beta} \right)^{1/\beta}, \tag{2.27}$$

then equation (1.1) has property (A).

Proof. It follows from (2.27) that

$$\lim_{n\to\infty} inf \left(\sum_{s=N}^{\infty} \frac{R_{s+m}^{\beta}}{R_s^{\beta}} p_s \right)^{1+1/\beta} c\frac{n^{1/\alpha}}{r_n^{1/\alpha}} \frac{R_n^{\beta}}{R_{n+m}^{\beta}} > \left(\frac{1}{\beta+1} \right)^{1+1/\beta} \left(\frac{\alpha}{\beta} \right)^{1/\beta} \tag{2.28}$$

where some $c \in (0,1)$. We set $\alpha = \frac{1}{\beta}$, $\gamma = \frac{1}{\alpha}$, $n + m = n$, $a_n = p_1^{-1/\beta}(n)$ and $b_n = c\frac{n^{1/\alpha}}{r_n^{1/\alpha}}$, $e_n = \sum_{s=n}^{\infty} p_1(s)$. The inequalities (2.16) and (2.24) reduce to inequalities (2.26) and (2.28). Then Corollary 2.1 ensure that equation (2.2) has no solution satisfying (2.3). From Theorem 2.1, we conclude that equation (1.1) has property (A).

∎

Theorem 2.5. Let $\alpha \geq \beta$, $\tau(n) = n - m$. Assume that (2.26) holds. If

$$\lim_{n\to\infty} inf \left(\sum_{s=n}^{\infty} p_s \right)^{1+1/\beta} \frac{n^{1/\alpha}}{r_n^{1/\alpha}} \frac{1}{p(s+m)} > \left(\frac{1}{\beta+1} \right)^{1+1/\beta} \left(\frac{\alpha}{\beta} \right)^{1/\beta}, \tag{2.29}$$

then equation (1.1) has property (A).

Proof. It follows from (2.29) that

$$\lim_{n\to\infty} inf \left(\sum_{s=n}^{\infty} p_s \right)^{1+1/\beta} c\frac{n^{1/\alpha}}{r_n^{1/\alpha}} \frac{1}{p(s+m)} > \left(\frac{1}{\beta+1} \right)^{1+1/\beta} \left(\frac{\alpha}{\beta} \right)^{1/\beta}$$

where some $c \in (0,1)$. We set $\alpha = \frac{1}{\beta}$, $\gamma = \frac{1}{\alpha}$, $n - m = n$, $a_n = p_2^{-1/\beta}(n)$ and $b_n = c\frac{n^{1/\alpha}}{r_n^{1/\alpha}}$, $e_n = \sum_{s=n}^{\infty} p(s)$. The inequality (2.24) reduce to inequality (2.29), therefore Corollary 2.1 ensure that inequality (2.10) has no solution satisfying (2.11). It follows from Theorem 2.2, we conclude that equation (1.1) has property (A).

∎

Corollary 2.2. Assume that equation (1.1) has property (A). If

$$\sum_{s=N}^{\infty} \frac{1}{r_s^{1/\alpha}} \left(\sum_{u=s}^{\infty} \sum_{j=u}^{\infty} p_j \right)^{1/\alpha} = \infty, \tag{2.30}$$

then every nonoscillatory solution of equation (1.1) tends to zero as $n \to \infty$.

Proof. The conclusion follows from the Definition 2.1 and equation (2.30), so it can be omitted.

□

3 Examples

In this section, we present two examples to illustrate the main results.

Example 3.1. Consider the third order nonlinear advanced difference equation

$$\Delta^2\big(n(\Delta x_n)\big) + \frac{4(n+3)^2}{(n+1)(n+2)} x_{n+3}^3 = 0\,, \qquad n \geq 1. \tag{3.1}$$

Here, $a_n = n$, $\alpha = 1$, $\beta = 3$, $\tau(n) = n + 3$ and $p_n = \frac{4(n+3)^2}{(n+1)(n+2)}$. It is easy to verify that condition (2.26) holds. From Theorem 2.4 we see that the equation (3.1) having property (A). In fact $\{x_n\} = \left\{\frac{1}{n}\right\}$ is one such solution of equation (3.1).

Example 3.2. Consider the third order nonlinear advanced difference equation

$$\Delta^2(2^n(\Delta x_n)^3) + \frac{9}{2^{-n+22}} x_{n-5}^3 = 0\,, \qquad n \geq 6. \tag{3.2}$$

Here, $a_n = 2^n$, $\alpha = 3$, $\beta = 3$, $\tau(n) = n - 5$ and $p_n = \frac{9}{2^{-n+22}}$. It is easy to verify that condition (2.29) holds. From Theorem 2.5 we see that the equation (3.2) having property (A). In fact $\{x_n\} = \left\{\frac{1}{2^n}\right\}$ is one such solution of equation (3.2).

We conclude the paper with the following remark.

Remark. It is easy to see that Examples 3.1 and 3.2 illustrate the Corollary 2.2.

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