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Oscillation criteria for certain even-order neutral delay difference equations

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Abstract

In this paper, we established sufficient conditions for the oscillation of solutions of even-order neutral type difference equation of the form

$$\Delta\left[r(n)\left(\Delta^{m-1}(x(n)+p(n)x(\tau(n)))\right)\right]+q(n)f(x(\sigma(n)))=0$$

under the conditions

$$\sum^{\infty}\frac{1}{r(n)}=\infty$$

and

$$\sum^{\infty}\frac{1}{r(n)}<\infty$$

Examples are also given to illustrate our results.

AMS subject classification: 39A10, 39A12.
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1. Introduction

The paper concerns the oscillatory behavior of solutions of higher order neutral type nonlinear difference equation of the following form

$$\Delta\left[r(n)\left(\Delta^{m-1}(x(n)+p(n)x(\tau(n)))\right)\right]+q(n)f(x(\sigma(n)))=0,\quad n\geq n_0 \tag{1.1}$$

where $n \geq 2$ and Δ is the forward difference operator defined by

$$\Delta x(n) = x(n+1) - x(n)$$

and

$$\Delta^i x(n) = \Delta(\Delta^{i-1}x(n))$$

and

$$n_0 \in N = \{n_0, n_0+1, n_0+2, ...\}$$

The following conditions are assumed to be hold:

(C_1) $\{p(n)\}$ and $\{q(n)\}$ are real sequences such that $0 \leq p(n) \leq p_0 < 1$ and $q(n) > 0$, where p_0 is a constant.

(C_2) $\{r(n)\}$ is a real sequence such that $r(n) > 0$ and $\Delta r(n) \geq 0$.

(C_3) $\{\tau(n)\}$ and $\{\sigma(n)\}$ are non negative integer sequences such that $\tau(n) < n, \sigma(n) < n$, $\Delta\sigma(n) > 0$, and $lim_{n\to\infty}\sigma(n) = lim_{n\to\infty}\tau(n) = \infty$.

(C_4) f is continuous $R \to R$ such that $\dfrac{f(x)}{x} \geq K > 0$, for $x \neq 0$ and K is a constant.

Further, We will consider the following two cases

$$\sum^{\infty}\frac{1}{r(n)} = \infty \tag{1.2}$$

$$\sum^{\infty}\frac{1}{r(n)} < \infty \tag{1.3}$$

Let $\mu = \max\{sup\{\tau(n), \sigma(n)\}\}$. Then by a solution of equation (1) we mean a real sequence $\{x(n)\}$ which is defined for $n \geq -\mu$ and which satisfies equation (1) for $n \geq 0$. A Solution $\{x(n)\}$ of equation (1) is said to be eventually positive if $x(n) > 0$ for all large n, and eventually negative if $x(n) < 0$ for all large n. It is said to be oscillatory if it is neither eventually positive nor eventually negative. We will also say that (1) is oscillatory if all of its solutions are oscillatory.

I. Kubiaczyk and S.H. Sekar [14] studied the second order sublinear delay difference equation

$$\Delta p_n(\Delta x_n) + q_n x_{n-\tau}^{\gamma} = 0, \quad 0 \leq \gamma \leq 1. \tag{E\_1}$$

By using the Riccati transformation techniques the authors obtained oscillation criteria for the equation (E_1) under the conditions

$$\sum_{n=n_0}^{\infty} \frac{1}{p_n} = \infty \text{ and } \sum_{n=n_0}^{\infty} \frac{1}{p_n} < \infty.$$

Pon. Sundaram and E. Thandapani [24] studied the oscillatory behavior of the solutions of the second order neutral difference equation

$$\Delta(L_1^{\alpha} x(n)) + f(n, xg(n))) = 0 \tag{E\_2}$$

where $L_1 x(n) = a(n)|\Delta L_0 x(n)|^{\alpha}$ sign $\Delta L_0(x(n))$ and $L_0(x(n)) = x(n) - p(n)x(\tau(n))$ under the conditions $0 \leq p(n) \leq 1$ and

$$\sum_{n=n_0}^{\infty} \frac{1}{(a(n))^{\frac{1}{\alpha}}} = \infty.$$

The authors obtained necessary and sufficient conditions for oscillation of almost all solution of (E_2) with $f(n, u)$ belonging to suitably restricted classes of superlinear and sublinear functions with respect to u.

In 1998, Wan-Tong Li [19] studied the oscillatory behavior of the following higher order nonlinear difference equation

$$\Delta(r_n(\Delta^{\alpha}(x_n - p_n x_{n-\tau}))^{\delta}) + f(n, x_{n-\sigma}) = 0, n \geq n_0 \tag{E\_3}$$

under the condition

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{r_n}\right)^{\frac{1}{\delta}} < \infty.$$

The author obtained some sufficient conditions which guarantee that every bounded solution of equation (E_3) is either oscillatory or tends to zero. These oscillation criteria, at least in some sense, complemented and improved those of X.Zhou and J.R. Yan [31].

In 2003, Z. Liu, S. Wu and Z. Zhang [20] considered the even order neutral difference equation

$$\Delta^{m-1}(a_n \Delta(x_n + \varphi(n, \tau_n) + q_n f(x(g_n)))) = 0 \tag{E\_4}$$

under the conditions $\sum_{n}^{\infty} \frac{1}{a_n} = \infty$ and $0 < \frac{\varphi(n, u)}{u} \leq p_n < 1$ for $u \neq 0$.

Some Riccati type difference inequalities are established for this equation (E_4) and the authors obtained some oscillation criteria using these difference inequalities.

Due to its numerous applications in fields such as economics and mathematical biology, the oscillation theory of difference equations has been receiving intensive attention. In the last two decades we refer the readers to the monographs [1, 2, 5, 8, 13, 16] and the references cited therein. The problem of finding sufficient conditions which ensure that all solutions of certain classes of even order neutral type difference equations are oscillatory has been studied by a number of authors and we refer the reader to the papers [3, 4, 6, 7, 9–12, 15, 17, 18, 21, 22, 25–30] and the references cited therein.

2. Auxiliary Lemmas

The following Lemmas will be needed in the proof of our main results.

Lemma 2.1. [1] (Discrete Kneser's Theorem) Let $\{u(n)\}$ be a sequence of real numbers in $N = \{0, 1, 2, \ldots\}$. Let $u(n) > 0$ and $\Delta^m u(n)$ be of constant sign with $\Delta^m u(n)$, not being identically zero on any subset $\{n_0, n_0+1, n_0+2, \ldots\}$. Then, there exists an integer l, $0 \le l \le m$ with $m + l$ odd for $\Delta^m u(n) \le 0$ and $m + l$ even for $\Delta^m u(n) \ge 0$ such that

$$l \le m - 1 \text{ implies } (-1)^{l+k}\Delta^k u(n) > 0, \text{ for all } n \in N, 1 \le k \le m - 1, \tag{2.1}$$

and

$$l \ge 1 \text{ implies } \Delta^k u(n) > 0, \text{ for all } n \in N, 1 \le k \le l - 1. \tag{2.2}$$

Lemma 2.2. [1] Let $z(n)$ be as defined in Lemma 2.1 and such that $z(n) > 0$ and $\Delta^m z(n) \le 0$ for all $n \ge n_0 \in N$. Then there exists a sufficiently large integer n_1, such that for all $n \ge n_1 \ge n_0$

$$z(n) \ge \frac{(n - n_1)^{(m-1)}}{(m - 1)!}\Delta^{m-1}z(2^{m-l-1}n) \tag{2.3}$$

where $(n - n_1)^{(m-1)}$ is the usual factorial notation. Moreover, if $\{z(n)\}$ is increasing, then

$$z(n) \ge \frac{(n)^{(m-1)}}{(m - 1)!}\Delta^{m-1}z(n)\frac{1}{(2^{m-1})^{(m-1)}} \quad \text{for all } n \ge 2n_1 \tag{2.4}$$

Lemma 2.3. [23] Assume that (2) holds. Furthermore, Assume that $x(n)$ is an eventually positive solution of (1). Then there exists $n \ge n_0$ such that

$$z(n) > 0, \Delta z(n) > 0, \Delta^{m-1}z(n) > 0 \text{ and } \Delta^m z(n) \le 0 \text{ for all } n \ge n_1 \tag{2.5}$$

3. Main Results

Theorem 3.1. Assume that (2) holds. If

$$\sum^{\infty} q(n) = \infty \tag{3.1}$$

then every solution $\{x(n)\}$ of equation (1) is oscillatory.

Proof. Let $x(n)$ be a nonoscillatry solution of (1). Without loss of generality, we may assume that $x(n)$ is eventually positive (The proof is similar when $x(n)$ is eventually negative).

That is, let $x(n) > 0$, $x(\tau(n)) > 0$ and let $x(\sigma(n)) > 0$ for $n \ge n_1 \ge 0$. Set

$$z(n) = x(n) + p(n)x(\tau(n)). \tag{3.2}$$

Since $p(n)$ is nonnegative, $z(n) > x(n) > 0$ for $n \geq n_1$. From (1) and (10), we have

$$\begin{aligned} \Delta\left(r(n)\Delta^{m-1}z(n)\right) &= r(n)\Delta^m z(n) + \Delta r(n)\Delta^{m-1}z(n) \\ &= -q(n)f(x(\sigma(n))) < 0. \end{aligned} \tag{3.3}$$

Then $r(n)\Delta^{m-1}z(n)$ is decreasing and $\Delta^{m-1}z(n)$ is eventually of one sign. Hence either

$$\Delta^{m-1}z(n) > 0 \;\text{ for }\; n \geq n_2 \geq n_1 \tag{3.4}$$

or

$$\Delta^{m-1}z(n) < 0 \;\text{ for }\; n \geq n_2 \geq n_1. \tag{3.5}$$

If (13) holds, then

$$r(n)\Delta^{m-1}z(n) \leq r(n_2)\Delta^{m-1}z(n_2) < 0. \;\text{ for }\; n \geq n_2. \tag{3.6}$$

Dividing this inequality by $r(n)$ and summing from n_2 to $n-1$, then by using (2), we get

$$\lim_{n\to\infty} \Delta^{m-1}z(n) = -\infty. \tag{3.7}$$

This result along with (13) leads to $\lim_{n\to\infty} z(n) = -\infty$.

But this contradicts the fact that $z(n) > 0$. Then (12) holds. Then from (11) and the fact that $r(n)$ is a positive nondecreasing sequence, we conclude that $\Delta^m z(n) < 0$, for $n \geq n_2$. It follows that $\Delta^i z(n)$ $(i = 0, 1, 2, \ldots, m-1)$ is strictly monotonic and of constant sign eventually.

By applying lemma 2.1, $z(n)$ satisfies (4) and (5), since m is even, the integer l associated with $z(n)$ is odd, that is $l \geq 1$. Hence, $z(n)$ is increasing for $n \geq n_3 \geq n_2$.

Then from (10) and the fact that $z(n)$ is increasing, we have

$$\begin{aligned} x(n) &= z(n) - p(n)x(\tau(n)) \\ &\geq z(n) - p(n)z(\tau(n)) \\ &\geq (1-p(n))z(n) \\ &\geq (1-p_0)z(n), \quad \text{for }\; n \geq n_3. \end{aligned} \tag{3.8}$$

Let $n_4 \geq n_3$ be such that $\sigma(n) \geq n_3$ for all $n \geq n_4$. Combining (C_4) and (16), we get

$$f(x(\sigma(n))) \geq K(1-p_0)z(\sigma(n)) \;\text{ for }\; n \geq n_4. \tag{3.9}$$

$$\geq K(1-p_0)z(\sigma(n+1))$$

In view of Lemma 2.2 and the decreasing character of $\Delta^{m-1}z(n)$, we have

$$\Delta z(\sigma(n)) \geq M(\sigma(n))^{(m-2)}\Delta^{m-1}z(n), \quad \text{for }\; n \geq n_5 \geq n_3. \tag{3.10}$$

Define

$$w(n) = \frac{r(n)\Delta^{m-1}z(n)}{z(\sigma(n))} \tag{3.11}$$

and then $w(n) > 0$ for $n \geq n_6 = max\{n_4, n_5\}$. By taking difference of *w(n)* and using (11), (17) and (18), we have

$$\begin{aligned}\Delta w(n) &= \frac{\Delta\left(r(n)\Delta^{m-1}z(n)\right)}{z(\sigma(n))} - \frac{r(n)\Delta^{m-1}z(n)\Delta z(\sigma(n))}{z(\sigma(n))z(\sigma(n+1))} \\ &\geq -K(1-p_0)q(n) - \frac{r(n)\Delta^{m-1}z(n)M(\sigma(n))^{(m-2)}\Delta^{m-1}z(n)}{z(\sigma(n))z(\sigma(n))} \\ &= -K(1-p_0)q(n) - \frac{M(\sigma(n))^{m-2}(w(n))^2}{r(n)} \end{aligned} \tag{3.12}$$

Since $r(n) > 0$, the term

$$\frac{M(\sigma(n))^{m-2}(w(n))^2}{r(n)} > 0 \tag{3.13}$$

Hence (20) reduces to

$$\Delta w(n) \leq -K(1-p_0)q(n). \tag{3.14}$$

Summing the inequality from n_7 to $n-1$, $n_7 > n_6$, and using the assumption (9), we see that $w(n) \to -\infty$ as $n \to \infty$. But this contradicts the positivity of $w(n)$. Hence the Theorem is proved. ■

In the above proof, being $l \geq 1$ plays an important role. In fact $l = 0$ is possible only for odd orders. In this case, the solutions are bounded. For unbounded solutions with m being odd, the integer l must be greater than or equal to 2. Thus, it is easy to show that if m is odd and the conditions of Theorem 3.1 are satisfied, then unbounded solution of (1) is oscillatory.

Remark 3.2. Theorem 3.1 remains true if the function f satisfies the condition that $xf(x) > 0$ and there exists a nondecreasing continuous function $\Phi : R - R$ with

$$|f(x)| \geq \Phi(|x|) \tag{3.15}$$

Remark 3.3. The condition (9) can be rewritten as

$$\sum^{\infty} q(n)[1 - p(\sigma(n))] = \infty \tag{3.16}$$

for the function $p(n)$. In this case, we consider unbounded solution.

Theorem 3.4. Assume that (3) and (9) holds and $n \geq 4$ is even. Further suppose that

$$\sum^{\infty}\left[Cq(s)(\sigma(s))^{m-2}\delta(s) - \frac{1}{4r(s)\delta(s)}\right] = \infty \tag{3.17}$$

where $C = \dfrac{\alpha K}{(m-2)!}$, $\alpha \in (0,1)$ is a constant, and $\delta(n) = \sum_{n}^{\infty} \dfrac{1}{r(s)}$. Then every solution $x(n)$ of (1) is either oscillatory or tends to zero as $n \to \infty$.

Proof. Assume that (1) has a nonoscillatory solution $x(n)$. Without loss of generality, we assume that there exists a $n_1 \geq n_2$ such that $x(n) > 0$, $x(\tau(n)) > 0$ and $x(\sigma(n)) > 0$ for all $n \geq n_1$.

Proceeding as in the proof of Theorem 3.1, we conclude that $r(n)\Delta^{m-1}z(n)$ is decreasing and $\Delta^{m-1}z(n)$ is eventually of one sign. Hence either (12) or (13) holds. If (12) holds, we obtain a contradiction by proceeding as in the proof of Theorem 3.1.

Suppose that (13) holds, that is $\Delta^{m-1}z(n) < 0$, for $n \geq n_2 \geq n_1$. Now we consider two assumptions: unbounded solutions and bounded solutions.

If the solution x(n) is unbounded, it is obvious that $z(n)$ is also unbounded. Since $z(n)\Delta^{m-1}z(n) < 0$ and $m-1$ is odd, we have by Lemma 2.1 that $l \geq 2$ (if $l = 0$, then $z(n)$ is bounded). Hence from (4), we have that $\Delta z(n) > 0$ and $\Delta^2 z(n) > 0$. Therefore $\lim_{n\to\infty} z(n) > 0$. Since $z(n)$ is increasing, we obtain

$$x(n) \geq (1 - p_0)z(n) \text{ for } n \geq n_3 \geq n_2. \tag{3.18}$$

By Lemma 2.2, and the fact that $\Delta^{m-2}z(n)$ is decreasing we get

$$z(\sigma(n)) \geq \frac{\lambda(\sigma(n))^{(m-2)}}{(m-2)!}\Delta^{m-2}z(n), n \geq n_4 \geq n_2. \tag{3.19}$$

Combining (C_4), (26) and (27), we obtain

$$f\,(x(\sigma(n))) \geq C(\sigma(n))^{(m-2)}\Delta^{m-2}z(n), \text{ for } n \geq n_5 = max\{n_3, n_4\},$$

where $C = \dfrac{\lambda K}{(m-2)!}$ with $\lambda = \left(\dfrac{1}{2^{(m-2)}}\right)^{(m-2)} \in (0,1)$ Define

$$w(n) = \frac{r(n)\Delta^{m-1}z(n)}{\Delta^{m-2}z(n)} \tag{3.20}$$

and then $w(n) < 0$ for $n \geq n_5$. Taking difference of $w(n)$ and using (11) and (28) we get

$$\begin{aligned} \Delta w(n) &\leq \frac{\Delta\left(r(n)\Delta^{m-1}z(n)\right)}{\Delta^{m-2}z(n+1)} - \frac{w^2(n)}{r(n)} \\ &\leq -Cq(n)(\sigma(n))^{(m-2)} - \frac{w^2(n)}{r(n)} \end{aligned} \tag{3.21}$$

As in the proof of Theorem (3.4) in case (1), we can show that

$$-1 \leq w(n)\delta(n) < 0. \tag{3.22}$$

Multiplying the inequality (30) by $\delta(n)$, and summing from n_5 to $n-1$ we get

$$w(n)\delta(n) - w(n_5)\delta(n_5) + \sum_{n_5}^{n-1}\frac{w(s)}{r(s)} + \sum_{n_5}^{n-1}\frac{w^2(s)\delta(s)}{r(s)} \leq -\sum_{n_5}^{n-1}Cq(s)(\sigma(s))^{(m-2)}\delta(s) \tag{3.23}$$

or

$$\begin{aligned} w(n)\delta(s) &- w(n_5)\delta(n_5) \\ &\leq -\sum_{n_5}^{n-1}\left[Cq(s)(\sigma(s))^{(m-2)}\delta(s) - \frac{1}{4r(s)\delta(s)}\right] - \sum_{n_5}^{n-1}\frac{\left[w(s)\delta(s)+\frac{1}{2}\right]^2}{r(s)\delta(s)} \end{aligned}$$

Thus

$$w(n)\delta(n) \leq -\sum_{n_5}^{n-1}\left[Cq(s)(\sigma(s))^{(m-2)}\delta(s) - \frac{1}{4r(s)\delta(s)}\right] + w(n_5)\delta(n_5) \tag{3.24}$$

Using assumption (25), we see that $w(n)\delta(n) \to -\infty$ as $n \to \infty$. But this contradicts (31).

In the solution x(n) is bounded, then z(n) is also bounded. Since $z(n)\Delta^{m-1}z(n) < 0$ and $m-1$ is odd, we have by Lemma 2.1 that $l = 0$ (otherwise, $z(n)$ is not bounded). Hence, from (4) and (5), we have

$$(-1)^i\Delta^i z(n) > 0, i = 0, 1, 2, \ldots, m-2 \text{ for } n \geq n_3 \geq n_2. \tag{3.25}$$

From (10) and the fact that $z(n) > x(n)$, we obtain

$$x(n) \geq z(n) - p_0x(\tau(n)) \geq z(n) - p_0z(\tau(n)) \tag{3.26}$$

or

$$x(n) \geq z(\tau(n))\left[\frac{z(n)}{z(\tau(n))} - p_0\right] \tag{3.27}$$

From (34), $z(n) > 0$, $\Delta z(n) < 0$ and $\Delta^2 z(n) > 0$, we have $\lim_{n\to\infty} z(n) = \lambda \geq 0$. Now we consider two cases.

Case I: Consider $\lambda > 0$. Since $z(n)$ is decreasing, there is an $\epsilon > 0$ and $n_4 \geq n_3$ such that, for $n \geq n_4$,

$$\lambda \leq z(n) \leq z(\tau(n)) \leq \lambda + \epsilon \tag{3.28}$$

From this, we can conclude that

$$\frac{z(n)}{z(\tau(n))} \geq \frac{\lambda}{\lambda+\epsilon} \tag{3.29}$$

Choose $n_5 \geq n_4$ such that, for $n \geq n_5$, we have $p_0 + \epsilon_1 \leq \dfrac{\lambda}{\lambda+\epsilon}$, for some $\epsilon_1 > 0$. Thus

$$\frac{z(n)}{z(\tau(n))} \geq p_0 + \epsilon, n \geq n_5. \tag{3.30}$$

Using this inequality in (36) and the fact that $z(n)$ is decreasing, we obtain

$$x(n) \geq \epsilon_1 z(n), n \geq n_5. \tag{3.31}$$

By Lemma 2.2, and the fact that $\Delta^{m-2}z(n)$ is decreasing, we obtain

$$z(\sigma(n)) \geq \frac{\lambda(\sigma(n))^{(m-2)}}{(m-2)!}\Delta^{m-2}z(n), n \geq n_6 \geq n_3. \tag{3.32}$$

Combining (C_4), (40) and (41), we obtain

$$f(x(\sigma(n))) \geq C(\sigma(n))^{(m-2)}\Delta^{m-2}z(n) \text{ for } n \geq n_7 = max\{n_5, n_6\},$$

where $C = \dfrac{\alpha K}{(m-2)!}$ with $\alpha = \epsilon_1\lambda \in (0, 1)$.

By using the transformation (29) and proceeding as in the previous assumption ($x(n)$ is unbounded), again we obtain a contradiction with (25).

Case II: Consider that $\lambda = 0$, since $x(n) \leq z(n)$, $x(n)$ tends to zero as $n \to \infty$, and this completes the proof. ■

4. Examples

In this section, we present two examples to illustrate the above results.

Example 4.1. Consider the following even order nonlinear neutral difference equation

$$\Delta\left[\sqrt{n}\Delta^{m-1}\left(x(n) + \frac{1}{2^2}x(n-2)\right)\right] + \frac{3^{m-1}}{2^{m+2}}(\sqrt{n+1} + 2\sqrt{n})x(n-3) = 0, n \geq 3 \tag{4.1}$$

Here

$$r(n) = \sqrt{n}, \quad p(n) = \frac{1}{2^2} < 1, \quad q(n) = \frac{3^{m-1}}{2^{m+2}}(\sqrt{n+1} + 2\sqrt{n})$$
$$\tau(n) = n-2, \ \sigma(n) = n-3, \ \Delta\sigma(n) > 0 \text{ and } \frac{f(x)}{x} = 1.$$

We can see that all conditions of Theorem 3.1 are satisfied. Thus, every solution of (43) is oscillatory. One such solution is $\{x(n)\} = \{\dfrac{(-1)^n}{2^n}\}$.

Example 4.2. Consider the following even order nonlinear neutral difference equations

$$\Delta\left[e^n\Delta^{m-1}\left(x(n) + \frac{1}{2^2}x(n-2)\right)\right] + \frac{1}{2^{n-3}}\frac{(e-2)e^n}{2^{m+2}} = 0, n \geq 3 \tag{4.2}$$

where

$$m \geq 4, \ p_0 = \frac{1}{2^2}, \ \tau(n) = n - 2, \ \sigma(n) = n - 3$$
$$q(n) = \frac{e^n(e-2)}{2^{m+2}} > 0$$

and $\frac{f(x)}{x} = 1$. It is easy to check that all conditions of Theorem 3.2 are satisfied. Thus, every solution of (44) is either oscillatory or tends to zero. Indeed one such solution $\{x(n)\} = \{\frac{1}{2^{n+1}}\}$ tends to zero as $n \to \infty$.

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