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Solution of Integro-Differential Equations by Using ELzaki Transform

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Abstract

In this paper, we solve the integro-differential equation by using new integral transform called ELzaki transform. New theorems for the transform of integrals are introduced and proved.

Keywords: ELzaki transform-Integro-differential equations.

Introduction

Many problems of physical interest are described by differential and integral equations with appropriate or boundary conditions. These problems are usually formulated as initial value problem, boundary value problems, or initial – boundary value problem that seem to be mathematically more vigorous and physically realistic in applied and engineering sciences. ELzaki transform method is very effective for solution of the response of differential and integral equations and a linear system of differential and integral equations.

The technique that we used is ELzaki transform method which is based on Fourier transform. It introduced by Tarig ELzaki (2010)

In this study, ELzaki transform is applied to integral and integrao-differential equations which the solution of these equations have a major role in the fields of science and engineering. When a physical system is modeled under the differential sense, if finally gives a differential equation, an integral equation or an integro-differential equation.

Recently .Tarig ELzaki introduced a new transform and named as ELzaki transform which is defined by:

$$\mathrm{E}\left[f(t),v\right]=T(v)=v\int_0^\infty e^{-\frac{t}{v}}f(t)\,dt \quad , \quad v\in(-k_1,k_2) \tag{1}$$

Or for a function $f\,(t)$ which is of exponential order,

$$\left|f\,(t)\right| < \begin{cases} Me^{-t/k_1} \quad , \quad t \le 0 \\ Me^{t/k_2} \quad , \quad t \ge 0 \end{cases} \tag{2}$$

ELzaki transform, henceforth designated by the operator $\mathrm{E}[.]$, is defined by the integral equation.

$$\mathrm{E}\left[f(t)\right] = T(v) = \quad v^2 \int_0^\infty f(vt)\; e^{-t} dt \;\;,\;\; -k_1 \le v \le k_2 \tag{3}$$

Where M is a real finite number and k_1, k_2 can be finite or finite.

Theorem (1-1):
Let $T\,(v)$ is the ELzaki transform of $f(t)$

$$\left[\mathrm{E}\left(f(t)\right) = T(v)\right] \quad and \quad g(t) = \begin{cases} f(t-\tau), & t \ge \tau \\ 0 \qquad\quad , & t < \tau \end{cases}$$

Then: $\mathrm{E}\left[g(t)\right] = e^{\frac{-\tau}{v}}\, T(v)$

Proof:

$$\mathrm{E}\left[g\,(t)\right] = \int_\tau^\infty v\, e^{-t/v} f\,(t-\tau) dt$$

. Let $t = \lambda + \tau$ we find that:

$$\int_0^\infty v e^{-(\frac{\lambda+\tau}{v})} f(\lambda)\, d\lambda = e^{-\tau/v} \int_0^\infty v\, e^{-\lambda/v} f(\lambda)\, d\lambda = e^{-\frac{\tau}{v}} T(v)$$

Which is the desired result

ELzaki transform can certainly treat all problems that are usually treated by the well- known and extensively used Laplace transform.

Indeed as the next theorem shows ELzaki transform is closely connected with the Laplace transform $F\,(s)$.

Theorem (1-2):
Let

$$f(t) \in A = \left\{ f(t) \middle| \exists M, k_1, k_2 > 0, such\ that \middle|, \left|f(t)\right| < Me^{|t|/k_i} \;,\quad if\; t \in (-1)^j \times [0, \infty) \right\}$$

With Laplace transform $F(s)$, Then ELzaki transform $T(v)$ of $f(t)$ is given by

$$T(v) = vF\left(\frac{1}{v}\right) \tag{4}$$

Proof:

Let: $f(t) \in A$. Then for $-k_1 < v < k_2$ $T(v) = v^2 \int_0^\infty e^{-t} f(vt)\,dt$

Let $w = vt$ then we have:

$$T(v) = v^2 \int_0^\infty e^{-\frac{w}{v}} f(w) \frac{dw}{v} = v \int_0^\infty e^{-\frac{w}{v}} f(w)\,dw = vF\left(\frac{1}{v}\right).$$

Also we have that $T(1) = F(1)$ so that both ELzaki and Laplace transforms must coincide at $v = s = 1$.

In fact the connection of ELzaki transform with Laplace transform goes much deeper, therefore the rules of F and T in (4) can be interchanged by the following corollary.

Corollary (1-3):

Let $f(t)$ having F and T for Laplace and ELzaki transforms respectively, then:

$$F(s) = sT\left(\frac{1}{s}\right) \tag{5}$$

Proof:

This relation can be obtained from (4) by taking $v = \frac{1}{s}$

The equations (4) and (5) form the duality relation governing these two transforms and may serve as a mean to get one from the other when needed.

ELzaki Transform of Derivatives and Integrals

Being restatement of the relation (4) will serve as our working definition, since the Laplace transform of $\sin t$ is $\frac{1}{1+s^2}$ then view of (4), its. ELzaki transform is $\mathrm{E}[\sin t] = \frac{v^3}{1+v^2}$ this exemplifies the duality between these two transforms.

Theorem (2-1):

Let $F'(s)$ and $T'(v)$ be the Laplace and ELzaki transforms of the derivative of $f(t)$.

Then:

$$(i)\ T'(v)=\frac{T(v)}{v}-v\,f(0) \tag{6}$$

$$(ii)\ T^{(n)}(v)=\frac{T(v)}{v^{n}}-\sum_{k=0}^{n-1} v^{2-n+k} f^{(k)}(0) \qquad , \quad n\geq 1 \tag{7}$$

Where $T^{n}(v)$ *and* $F^{n}(s)$ are ELzaki and Laplace transforms of the nth derivative $f^{(n)}(t)$ of the function $f(t)$.

Proof:

(i) Since the Laplace transform of the derivatives of $f(t)$ is $F'(s)=sF(s)-f(0)$ then:

$$T'(v)=v\,F'\left(\frac{1}{v}\right)=v\left[\frac{1}{v}F\left(\frac{1}{v}\right)-f(0)\right]=F\left(\frac{1}{v}\right)-v\,f(0)=\frac{T(v)}{v}-v\,f(0)$$

(ii) By definition, the Laplace transform for $f^{(n)}(t)$ is given by

$$F^{(n)}(s)=S^{n}F(s)-\sum_{k=0}^{n-1}S^{n-(k+1)}f^{(k)}(0)$$

Therefore:

$$F\left(\frac{1}{v}\right)=\frac{F\left(\frac{1}{v}\right)}{v^{n}}-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{v^{n-(k+1)}}$$

Now, since

$$T^{(k)}(v)=v\ F^{(k)}\left(\frac{1}{v}\right) \quad for \qquad 0\leq k\leq m\ ,\ \ we\ \ have\ T^{(n)}(v)=\frac{T(v)}{v^{n}}-\sum_{k=0}^{n-1}v^{2-n+k}f^{(k)}(0)$$

Theorem (2-2)

Let $T'(v)$ *and* $F'(s)$ denote ELzaki and the Laplace transforms of the definite integral of $f(t)$.

$$h(t)=\int_{0}^{t}f(\tau)\,d\tau. \qquad Then \quad T'(v)=E\left[h(t)\right]=vT(v)$$

Proof:

By the definition of Laplace transform $F'(s)=L\big(h(t)\big)=\dfrac{F(s)}{s}$

Hence:

$$T'(v) = vF'\left(\frac{1}{v}\right) = v\left[vF\left(\frac{1}{v}\right)\right] = v^2 F\left(\frac{1}{v}\right) = vT(v)$$

Theorem (2-3) (shift):
Let $f(t) \in A$ with ELzaki transform $T(v)$.Then:

$$E\left[e^{at} f(t)\right] = \frac{1}{1-av} \quad T\left[\frac{v}{1-av}\right]$$

Proof:
From definition of ELzaki transform we have:

$$E\left[e^{at} f(t)\right] = v^2 \int_0^\infty f(vt) e^{-(1-av)t} dt \text{ Let } w = (1-av)t \Rightarrow dw = (1-av)dt,$$

Then:

$$\frac{v^2}{1-av}\int_0^\infty f\left[\frac{wv}{1-av}\right] e^{-w} dw = \frac{1}{1-av} \quad T\left[\frac{v}{1-av}\right]$$

Theorem (2-4) (convolution):
Let $f(t)$ *and* $g(t)$ be defined in A having Laplace transforms $F(s)$ and $G(s)$ and ELzaki transforms $M(v)$ and $N(v)$. Then ELzaki transform of the Convolution of f and g

$$(f*g)(t) = \int_0^\infty f(t) g(t-\tau) d\tau \text{ Is given by: } E\left[(f*g)(t)\right] = \frac{1}{v} M(v) N(v)$$

Proof:
The Laplace transform of $(f*g)$ is given by: $L\left[(f*g)\right] = F(s)G(s)$

By the duality relation (4) we have: $E\left[(f*g)(t)\right] = vL\left[(f*g)(t)\right]$, and since

$$M(v) = vF\left(\frac{1}{v}\right),\ N(v) = vG\left(\frac{1}{v}\right) \quad \text{Then} \quad E\left[(f*g)(t)\right] = v\left[F\left(\frac{1}{v}\right).G\left(\frac{1}{v}\right)\right] =$$

$$v\left[\frac{M(v)}{v} \cdot \frac{N(v)}{v}\right] = \frac{1}{v} M(v) N(v)$$

Applications to Integral Equations and Integro-Differential Equations

Integral equation is an equation having the form.

$$y(t) = F(t) + \int_a^b k(u,t) \quad y(u) du$$

Where $F(t)$ and $K(u,t)$ are known, a,b are either given constants or function of t, and the function $y(t)$ which appears under the integral sign is to be determined. The function $k(u,t)$ is often called the kernel of the integral equation. If a and b are constants, the equation is often called a Fredholm integral equation. If a is constant while $b=t$, it is called a Volterra integral equation.

In this work, we show how ELzaki transform method can be applied successfully to solve, the condition integral equations.

This method is simple and straight for ward, and can be illustrated by examples:

To solve the convolution integral equation of the form:

$$f(t)=h(t)+\lambda\int_0^t g(t-\tau)f(\tau)d\tau \tag{8}$$

We take the ELzaki transform of this equation to obtain.

$$T(v)=\bar{h}(v)+\lambda \mathrm{E}\left\{\int_0^t g(t-\tau)f(\tau)d\tau\right\}$$

Which is, by the convolution theorem we have:

$$T(v)=\bar{h}(v)+\frac{\lambda}{v}T(v)\bar{g}(v)$$

Where that $T(v)$ is Elzaki transform of the functions $f(t)$,then:

$$T(v)=\frac{v\bar{h}(v)}{v-\lambda\bar{g}(v)}$$

Inversion gives the formal solution in the form:

$$f(t)=T^{-1}\left[\frac{v\bar{h}(v)}{v-\lambda\bar{g}(v)}\right]$$

In many simple cases, the right-hand side can be inverted by using partial fraction or any method .Hence, the solution can readily be found.

Example (1)

Consider the following linear integral equation, which is solved by ELzaki transform.

$$y(t)=t+\frac{1}{6}\int_0^t (t-u)^3 y(u)du \tag{9}$$

Applying ELzaki transform of this equation we have

$$T(v) = v^3 + \frac{1}{6v}\left[6v^5\, T(v)\right] = v^3 + v^4\, T(v)$$

Or

$$T(v) = \frac{v^3}{1-v^4} = \frac{1}{2}\left[\frac{v^3}{1+v^2} + \frac{v^3}{1-v^2}\right] \quad (10)$$

By using inverse ELzaki transform of equation (10) we obtain the solution

$$y(t) = \frac{1}{2}\sin t + \frac{1}{2}\sin \mathrm{h}\, t$$

Example (2)
Consider the following boundary value type integro-differential equation

$$\int_0^t y'(u)\, y(t-u)\, du = 24t^3 \quad , \quad y(0) = 0 \quad (11)$$

Being in a similar way with the first example, we obtain

$$\frac{1}{v}\left[\frac{T(v)}{v} .\, T(v)\right] = 144\; v^5$$

$$T^2(v) = \;\; 144\; v^5 \qquad or \quad T(v) = \;\pm 12\;\; v^{\frac{5}{2}} \quad (12)$$

Inverting equation (12) and using the relation $\mathrm{E}\left[t^a\right] = \Gamma(a+1)v^{\,a+2}$

We obtain the solution in the form $y(t) = \dfrac{\pm 12\;\; t^{3/2}}{\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}} = \pm\dfrac{16\; t^{3/2}}{\sqrt{\pi}}$

Example (3)
Solve the integro-differential equation

$$f'(t) = \delta(t) + \int_0^t f(\tau)\cos(t-\tau)\, d\tau,\; f(0) = 1 \quad (13)$$

Where $\delta(t)$ is a direct delta function. On using ELzaki transform for eq (13) we have

$$\frac{T(v)}{v} \;-\; vf(0) = v + \frac{1}{v}\left[T(v)\left(\frac{v^2}{1+v^2}\right)\right]$$

$$T(v)\left[1 - \frac{v^2}{1+v^2}\right] = 2v^2 \;\Rightarrow\; T(v) = 2v^2 + 2v^4$$

Apply the inverse ELzaki transform to find the Solution of (13) in the form

$$f\left(t\right)= 2+t^{2}$$

Example (4)

Find the solution of the integral equation:

$$f\left(t\right)=a\,t^{n}-e^{-bt}-c\int_{0}^{t} f\left(\tau\right)e^{c(t-\tau)}d\tau \tag{14}$$

Where a,b,c are constants.

Solution:

Taking ELzaki transform, we obtain

$$T\left(v\right)=a\,n!v^{n+2}-\frac{v^{2}}{1+bv}-\frac{c}{v}\left[T\left(v\right).\frac{v^{2}}{1-cv}\right]$$

$$\frac{T\left(v\right)}{1-cv}=a\,n!v^{n+2}-\frac{v^{2}}{1+bv} \text{ And } T\left(v\right)=a\,n!v^{n+2}-c\,a\,n!v^{n+3}-\frac{v^{2}}{1+bv}+\frac{cv^{3}}{1-cv}$$

$$T\left(v\right)=a\,n!v^{n+2}-c\,a\,n!v^{n+3}-\frac{v^{2}}{1+bv}-\frac{cv^{2}}{b}-\frac{\frac{c}{b}v^{2}}{1-cv}$$

Inversion yields the solution of (14)

$$f\left(t\right)=a\,t^{n}-\frac{c\,a\,n!}{\left(n+1\right)!}t^{n+1}-\frac{c}{b}-\left(1+\frac{c}{b}\right)e^{-bt}$$

Example (5)

Solve the following linear integro-differential equation.

$$y^{(iv)}=x+x\,e^{x}+3e^{x}+y\left(x\right)-\int_{0}^{x} y\left(t\right)\, dt \tag{15}$$

With the initial Conditions

$$y\left(0\right)=1 \quad , \quad y'\left(0\right)=1 \quad , \quad y''\left(0\right)=2 \quad , \quad y'''\left(0\right)=3 \tag{16}$$

By using ELzaki transform of Eq (15) we have:

$$\frac{T\left(v\right)}{v^{4}}-\frac{y\left(0\right)}{v^{2}}-\frac{1}{v}y'\left(0\right)-y''\left(0\right)-vy'''\left(0\right)=v^{3}+\frac{v^{3}}{\left(1-v\right)^{2}}+\frac{3v^{2}}{1-v}+T\left(v\right)-vT\left(v\right) \tag{17}$$

Substituting Eq (16) into Eq(15) we get:

$$\left(1+v^5-v^4\right)T\left(v\right)= \quad v^2+v^3+2v^4+3\,v^5+v^7+\frac{v^7}{\left(1-v\right)^2}\;+\frac{3\,v^6}{1-v}$$

$$T\left(v\right)=\;v^2+\frac{v^3}{\left(1-v\right)^2}$$

Then by using inverse ELzaki transform we obtain the Solution in the form of

$$y\left(x\right)=\quad 1+xe^x$$

Example (6)

Consider the following integro-differential equation.

$$f\left(t\right)=a\sin t+2\int_0^t f'\left(\tau\right)\sin\left(t-\tau\right)\;d\tau\;,\quad f(0)=0 \qquad (18)$$

Taking ELzaki transform of Eq (18) we get:

$$T\left(v\right)=\;\frac{av^3}{1+v^2}\;+\;\frac{2}{v}\left[\left(\frac{T\left(v\right)}{v}-v\,f\;\left(0\right)\right)\left(\frac{v^3}{1+v^2}\right)\right]$$

Using the condition $f\;\left(0\right)=0$ to obtain:

$$T\left(v\right)\left[1-\frac{2v}{1+v^2}\right]=\frac{a\,v^3}{1+v^2}\;\text{ Or }T\;\left(v\right)=\frac{av^3}{\left(1-v\right)^2}$$

Inversion yields the solution of (18) in the form: $f\left(t\right)=\;a\,t\,e^t$

Conclusion

In this study we introduced new integral transform to solve the Integro-differential equations. It is shown that ELzaki transform is a very efficient tool for solving Integro-differential equation in the bounded domains.

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