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Sufficiency and Duality in Multiobjective Variational Control Problems with Generalized ρ-Univex Type I Functions

[1]Raman Patel and [2]P.M. Patel

[1]Department of Statistics, Veer Narmad South Gujarat University, Surat–395007, India
[2]Department of Statistics, Sir K.P. College of Commerce, Athwalines, Surat, India

Abstract

The concept of the class of ρ-V-univex type I functions and their generalizations are introduced. Using these functions, sufficient optimality conditions and mixed type duality results are obtained for multiobjective variational control problems.

Keywords: Multiobjective programming, Variational problems, Mixed type duality, generalized ρ-V-univex type I functions.

AMS Subject Classification: 90C29, 90C32, 49J40.

Introduction

Convexity plays a vital role in many aspects of mathematical programming including sufficient optimality conditions and duality theory. To relax convexity assumptions imposed on the functions involved, various generalized notions have been proposed. One of the useful generalizations is generalized (F,ρ)-convexity introduced by Preda [21], an extension of F-convexity defined by Hanson and Mond [9] and generalized ρ-convexity defined by Vial [23,24].

Hanson and Mond [8] considered a dual formulation for a class of variational problems. Mond and Hanson [13] have obtained duality results for control problems. Mishra and Mukherjee [11] discussed duality for multiobjective variational problems containing generalized (F,ρ)-convex functions. Some duality results for a class of differentiable multiobjective variational problems were studied in [4]. Mukherjee and Rao [15] considered a mixed type dual for multiobjective variational problem and various duality results were established by relating efficient solutions between this mixed type dual pair. Ahmad and Gulati [2] considered a mixed type duality model for multiobjective variational problems and a number of duality results were established by relating proper efficient solutions between this mixed type dual pair. Nahak and Nanda [16] used the concept of efficiency to formulate Wolfe and Mond-Weir type duals for multiobjective variational control problems and established weak and strong duality theorems under generalized (F,ρ)-convexity assumptions. Patel [19] used the concept of efficiency to formulate Wolfe and Mond-Weir type duals for multiobjective fractional variational control problems and established weak and strong duality theorems under generalized (F,ρ)-convexity assumptions.

Bector and Singh [3] introduced a class of functions called b-vex functions. Pandian [18] defined (b,F,ρ)-convex functions and established duality results for multiobjective programming problems. Bhatia and Kumar [5] introduced b-vex functions for variational problems and established some duality results. Bhatia and Mehra [6] introduced B-type I functions and generalized B-type I functions for continuous case. Bhatia and Sharma [7] introduced BF-type I functions and their generalizations for continuous case and established optimality and duality results. Mishra et al. [12] introduced the class of V-univex type I functions and their generalizations. Khazafi and Rueda [10] extended V-univex type I functions for multiobjective variational programming problems and various sufficiency and mixed type duality results were established under generalized V-univex type I functions.

In this paper, we extend the class of V-univex type I functions and their generalizations to multiobjective variational control problems on the lines of Khazafi and Rueda [10] and obtain sufficiency and mixed type duality results for multiobjective variational control problems.

Definitions and Preliminaries

We use the following notations for vector inequalities. For $x,y \in R^n$, we have

$$x \le y \text{ iff } x_i \le y_i,\ i=1,2,\dots,n,$$

$$x < y \text{ iff } x_i < y_i,\ i=1,2,\dots,n.$$

Let I=[a,b] be real interval and $K=\{1,2,...,k\}$, $M=\{1,2,...,m\}$. Let $\phi : IxR^n xR^n xR^m xR^m \to R$ be continuously differentiable function. In order to consider $\phi(t,x(t),\dot{x}(t),u(t),\dot{u}(t))$, where $x(t){:}I \to R^n$, $u(t){:}I \to R^m$ are differentiable with derivatives $\dot{x}(t)$ and $\dot{u}(t)$ respectively. For notational simplicity, we write $x(t),\dot{x}(t),u(t),\dot{u}(t)$ as $x,\dot{x},u,\dot{u}$ respectively, as and when necessary. We denote the partial derivatives of ϕ by ϕ_x and $\phi_{\dot{x}}$, where

$$\phi_x = \left[\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \dots, \frac{\partial \phi}{\partial x_n}\right],$$

$$\phi_{\dot{x}} = \left[\frac{\partial\phi}{\partial\dot{x}_1}, \frac{\partial\phi}{\partial\dot{x}_2}, \ldots, \frac{\partial\phi}{\partial\dot{x}_n}\right].$$

The partial derivatives of other functions used will be written similarly. Let $PS(I,R^n)$ denote the space of all piecewise smooth n-dimensional vector functions x defined on compact subset I of R with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differential operator D is given by

$$y = Dx \iff x(t) = \alpha + \int_a^b y(s)ds$$

in which α is a given boundary value. Therefore $D = \frac{d}{dt}$ except at discontinuties.

We consider the following multiobjective variational control problem:

$$\text{(MOP)}\quad \text{Minimize} \int_a^b f(t,x,\dot{x},u,\dot{u})dt = \left[\int_a^b f^1(t,x,\dot{x},u,\dot{u})dt, \ldots, \int_a^b f^k(t,x,\dot{x},u,\dot{u})dt\right],$$

subject to $x(a) = \alpha$, $x(b) = \beta$, $h^j(t,x,\dot{x},u,\dot{u}) \le 0$, $t \in I$, $j \in \{1,2,..,m\}$, f_i, $i \in K = \{1,2,\ldots,k\}$, and h_j, $j \in M = \{1,2,\ldots,m\}$, are assumed to be continuously differentiable functions defined on $IxR^nxR^nxR^mxR^m$. Let A be the set of feasible solutions of (MOP). Efficiency and proper efficiency are defined in their usual sense as defined in [4].

Definition 2.1: A functional $F:IxR^nxR^nxR^mxR^mxR^nxR^nxR^mxR^m \to R$ is said to be sublinear if for any x, $x^0 \in R^n$, $\dot{x},\dot{x}^0 \in R^n$, u, $u^0 \in R^m$, $\dot{u},\dot{u}^0 \in R^m$,

$$F\left[t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0;\alpha_1+\alpha_2\right] \le F\left[t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0;\alpha_1\right]$$
$$+ F\left[t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0;\alpha_2\right], \quad \text{for any } \alpha_1,\alpha_2 \in R^n, \qquad \text{(A)}$$

and

$$F\left[t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0;\alpha a\right] = \alpha F\left[t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0;a\right],$$
$$\text{for any } \alpha \in R,\ \alpha \ge 0,\ a \in R^n. \qquad \text{(B)}$$

It follows from (A) and (B) that

$$F\left[t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0;0\right] = 0.$$

We define the following univex type I functions and their generalizations.

Let us consider a sublinear functional F and the functional $f,h:IxR^nxR^nxR^mxR^m \to R$. We assume that f and h are continuously differentiable functions. Let

ϕ_0:$R^k \to R^k$, ϕ_1:$R^m \to R^m$, b_0, b_1: $PS(I,R^n) \times PS(I,R^m) \times PS(I,R^n) \times PS(I,R^m) \to R_+$ and η:$I\times R^n \times R^m \times R^n \times R^m \to R^n$, $\rho=(\rho^1,\rho^2)$, where $\rho^1=(\rho_1,\rho_2,\ldots,\rho_k)\in R^k$, $\rho^2=(\rho_{1+k},\rho_{2+k},\ldots,\rho_{m+k})\in R^m$, $d(t,.,.,.,.)$ be pseudometric on R^n.

Definition 2.2: A pair (f,h) is said to be ρ-V-univex type I at $x^0\in PS(I,R^n)$, $u^0\in PS(I,R^m)$ with respect to ϕ_0, ϕ_1, b_0, b_1, η such that for all $(x,u)\in A$, we have

$$b_0(x,u,x^0,u^0)\phi_0\left[\int_a^b f(t,x,\dot{x},u,\dot{u})dt - \int_a^b f(t,x^0,\dot{x}^0,u^0,\dot{u}^0)dt\right]$$

$$\geqq \int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^t\left(f_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}f_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)\right)\end{array}\right]dt + \rho^1 d^2(x,u,x^0,u^0), \tag{2.1}$$

$$- b_1(x,u,x^0,u^0)\phi_1\int_a^b h(t,x^0,u^0,\dot{x}^0,\dot{u}^0)dt$$

$$\geqq \int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^t\left(h_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}h_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)\right)\end{array}\right]dt + \rho^2 d^2(x,u,x^0,u^0). \tag{2.2}$$

If (2.1) is satisfied as a strict inequality then we say that a pair (f,h) is semi-strictly ρ-V-univex type I at (x^0,u^0) with respect to ϕ_0, ϕ_1, b_0, b_1, η.

Remark

i. When $\rho^1,\rho^2 = 0$, ϕ_0, $\phi_1=1$, the concept of (b,F,ρ)-type I is the same as that of BF-type I in Ref. 7.

ii. When $\eta(t,x,u,x^0,u^0)=1$, $\rho^1,\rho^2 = 0$, ϕ_0, $\phi_1=1$, the same concept appeared in the definition of (b,F)-convex in Ref. 17.

iii. When ϕ_0, $\phi_1=1$, the concept of (b,F,ρ)-type I is the same as that of (b,F,ρ)-type I in Ref. 20.

Definition 2.3: A pair (f,h) is said to be weakly ρ-V-strictly pseudoquasi univex type I at $x^0\in PS(I,R^n)$, $u^0\in PS(I,R^m)$ with respect to ϕ_0, ϕ_1, b_0, b_1, η such that for all $(x,u)\in A$, we have

$$\phi_0 \int_a^b f(t,x,\dot{x},u,\dot{u})dt \;\; \leq \;\; \phi_0 \int_a^b f(t,x^0,\dot{x}^0,u^0,\dot{u}^0)dt$$

$$\Rightarrow \; b_0(x,u,x^0,u^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^t(f_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}f_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)) \end{array}\right]dt$$

$$< -\rho^1 d^2(x,u,x^0,u^0),$$

$$-\phi_1 \int_a^b h(t,x^0,u^0,\dot{x}^0,\dot{u}^0)dt \leqq 0$$

$$\Rightarrow \; b_1(x,u,x^0,u^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^t(h_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}h_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)) \end{array}\right]dt$$

$$\leqq -\rho^2 d^2(x,u,x^0,u^0).$$

Definition 2.4: A pair (f,h) is said to be strongly ρ-V-pseudoquasi univex type I at $x^0 \in PS(I,R^n)$, $u^0 \in PS(I,R^m)$ with respect to ϕ_0, ϕ_1, b_0, b_1, η such that for all $(x,u) \in A$, we have

$$\phi_0 \int_a^b f(t,x,\dot{x},u,\dot{u})dt \;\; \leq \;\; \phi_0 \int_a^b f(t,x^0,\dot{x}^0,u^0,\dot{u}^0)dt$$

$$\Rightarrow \; b_0(x,u,x^0,u^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^t(f_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}f_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)) \end{array}\right]dt$$

$$\leq -\rho^1 d^2(x,u,x^0,u^0),$$

$$-\phi_1 \int_a^b h(t,x^0,u^0,\dot{x}^0,\dot{u}^0)dt \leqq 0$$

$$\Rightarrow \; b_1(x,u,x^0,u^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^t(h_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}h_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)) \end{array}\right]dt$$

$$\leqq -\rho^2 d^2(x,u,x^0,u^0).$$

Definition 2.5: A pair (f,h) is said to be weakly ρ-V-strictly pseudo univex type I at $x^0 \in PS(I,R^n)$, $u^0 \in PS(I,R^m)$ with respect to ϕ_0, ϕ_1, b_0, b_1, η such that for all $(x,u) \in A$, we have

$$\phi_0\int_a^b f(t,x,\dot{x},u,\dot{u})dt \leq \phi_0\int_a^b f(t,x^0,\dot{x}^0,u^0,\dot{u}^0)dt$$

$$\Rightarrow b_0(x,u,x^0,u^0)\int_a^b F\begin{bmatrix} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^t(f_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}f_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)) \end{bmatrix}dt$$

$$< -\rho^1 d^2(x,u,x^0,u^0),$$

$$-\phi_1\int_a^b h(t,x^0,u^0,\dot{x}^0,\dot{u}^0)dt \leqq 0$$

$$\Rightarrow b_1(x,u,x^0,u^0)\int_a^b F\begin{bmatrix} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^t(h_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}h_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)) \end{bmatrix}dt$$

$$< -\rho^2 d^2(x,u,x^0,u^0).$$

Sufficient Conditions

Under generalized ρ-V-univexity type I conditions, in this section, we establish various sufficient optimality conditions for (MOP).

Theorem 3.1: Assume that (x^0,u^0) is a feasible solution for (MOP) and assume that there exists $\lambda^0 \in R^k$, $\lambda^0 \geq 0$, $\beta^0 \in PS(I,R^m_+)$ such that the following relations hold for all $t \in I$:

$$\lambda^{0T}f_x(t,x^0,\dot{x}^0,u^0,\dot{u}^0) + \beta^0(t)^T h_x(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - \frac{d}{dt}\left[\lambda^{0T}f_{\dot{x}}(t,x^0,\dot{x}^0,u^0,\dot{u}^0) + \beta^0(t)^T h_{\dot{x}}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\right] = 0, \tag{3.1}$$

$$\beta^0(t)^T h(t,x^0,\dot{x}^0,u^0,\dot{u}^0) = 0, \tag{3.2}$$

$$\beta^0(t) \geqq 0, \quad t \in I. \tag{3.3}$$

Further, assume that $(f,\beta^0(t)^T h)$ is strongly ρ-V-pseudoquasi univex type I at (x^0,u^0) with respect to functions ϕ_0, ϕ_1, b_0,b_1,η with $b_1(x,u,x^0,u^0)>0$ for all $(x,u) \in A$. Also suppose that $\phi_1(0) \geqq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, provided $\lambda^{0T}\rho^1 + \beta^{0T}\rho^2 \geq 0$, then (x^0,u^0) is an efficient solution for (MOP).

Proof: If (x^0,u^0) is not an efficient solution for (MOP), then there exists $(x,u) \in A$ such that $\int_a^b f(t,x,\dot{x},u,\dot{u})dt \leq \int_a^b f(t,x^0,\dot{x}^0,u^0,\dot{u}^0)dt$.

From (3.2), we have

$$\int_a^b \beta^0(t)^T h(t,x^0,\dot{x}^0,u^0,\dot{u}^0)dt = 0.$$

Using $\phi_1(0) \geqq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, we get

$$\phi_0\left[\int_a^b f(t,x,\dot{x},u,\dot{u})dt - \int_a^b f(t,x^0,\dot{x}^0,u^0,\dot{u}^0)dt\right] \leq 0, \tag{3.4}$$

$$-\phi_1\left[\int_a^b \beta^0(t)^T h(t,x^0,\dot{x}^0,u^0,\dot{u}^0)dt\right] \leqq 0. \tag{3.5}$$

Since $(f,\beta^0(t)^t h)$ is strongly ρ-V-pseudoquasi univex type I at (x^0,u^0) with respect to ϕ_0, ϕ_1, b_0,b_1,η,

$$b_0(x,u,x^0,u^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^T(f_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}f_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)) \end{array}\right]dt$$

$$\leq -\rho^1 d^2(x,u,x^0,u^0),$$

$$b_1(x,u,x^0,u^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^T(\beta^0(t)^T h_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}\beta^0(t)^T h_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)) \end{array}\right]dt$$

$$\leqq -(\beta^{0T}\rho^2)d^2(x,u,x^0,u^0).$$

Since $b_1(x,u,x^0,u^0) > 0$ and $\lambda^{0T} > 0$, we get

$$b_0(x,u,x^0,u^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^T(\lambda^{0T}f_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}(\lambda^{0T}f_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0))) \end{array}\right]dt \tag{3.6}$$

$$< -(\lambda^{0T}\rho^1)d^2(x,u,x^0,u^0),$$

$$\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^T(\beta^0(t)^T h_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}\beta^0(t)^T h_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)) \end{array}\right]dt \tag{3.7}$$

$$\leqq -(\beta^{0T}\rho^2)d^2(x,u,x^0,u^0).$$

Since $b_0(x,u,x^0,u^0) \geqq 0$, it follows that

$$b_0(x,u,x^0,u^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^T(\beta^0(t)^T h_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}\beta^0(t)^T h_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)) \end{array}\right]dt \tag{3.8}$$
$$\leqq -(\beta^{0T}\rho^2)d^2(x,u,x^0,u^0).$$

Adding (3.6) and (3.8), we get

$$b_0(x,u,x^0,u^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^T(\lambda^{0T}f_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) + \beta^0(t)^T h_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) \\ -\frac{d}{dt}(\lambda^{0T}f_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0) + \beta^0(t)^T h_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0))) \end{array}\right]dt$$
$$< -(\lambda^{0T}\rho^1 + \beta^{0T}\rho^2)d^2(x,u,x^0,u^0),$$

which contradicts (3.1). Hence (x^0,u^0) is an efficient solution for (MOP) and the proof is complete.

In the next theorem, we replace strongly ρ-V-pseudoquasi univex type I by weakly ρ-V-strictly pseudoquasi univex type I of $(f,\beta^0(t)^t h)$.

Theorem 3.2: Assume that $(x^0,u^0)\in A$ is a feasible solution for (MOP) and there exists $\lambda^0 \in R^k$, $\lambda^0 \geq 0$, $\beta^0 \in PS(I,R^m_+)$ such that (3.1)-(3.3) of theorem 3.1 are satisfied.

Further, assume that $(f,\beta^0(t)^T h)$ is weakly ρ-V-strictly pseudoquasi univex type I at (x^0,u^0) with respect to functions ϕ_0, ϕ_1, b_0,b_1,η with $b_1(x,u,x^0,u^0)>0$ for all $(x,u)\in A$. Suppose that $\phi_1(0) \geqq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, provided $\lambda^{0T}\rho^1 + \beta^{0T}\rho^2 \geq 0$, then (x^0,u^0) is an efficient solution for (MOP).

Proof: If (x^0,u^0) is not an efficient solution for (MOP), then there exists $(x,u)\in A$ such that $\int_a^b f(t,x,\dot{x},u,\dot{u})dt \leq \int_a^b f(t,x^0,\dot{x}^0,u^0,\dot{u}^0)dt.$

From (3.2), we have $\int_a^b \beta^0(t)^T h(t,x^0,\dot{x}^0,u^0,\dot{u}^0)dt = 0.$

Using $\phi_1(0) \geqq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, we get

$$\phi_0\left[\int_a^b f(t,x,\dot{x},u,\dot{u})dt - \int_a^b f(t,x^0,\dot{x}^0,u^0,\dot{u}^0)dt\right] \leq 0,$$

$$-\phi_1\left[\int_a^b \beta^0(t)^T h(t,x^0,\dot{x}^0,u^0,\dot{u}^0)dt\right] \leqq 0.$$

Since $(f,\beta^0(t)^t h)$ is weakly ρ-V-strictly pseudoquasi univex type I at (x^0,u^0) with respect to ϕ_0, ϕ_1, b_0,b_1,η, so

$$b_0(x,u,x^0,u^0)\int_a^b F\left[\begin{matrix} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^T(f_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}f_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)) \end{matrix}\right]dt$$

$$< -\rho^1 d^2(x,u,x^0,u^0),$$

$$b_1(x,u,x^0,u^0)\int_a^b F\left[\begin{matrix} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^T(\beta^0(t)^T h_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}\beta^0(t)^T h_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)) \end{matrix}\right]dt$$

$$\leqq -(\beta^{0T}\rho^2)d^2(x,u,x^0,u^0).$$

Remaining part of the proof follows on similar lines as that of theorem 3.1.

In the final sufficiency result below, we invoke the weak ρ-V-strictly pseudo univex type I of $(f,\beta^0(t)^T h)$.

Theorem 3.3: Assume that $(x^0,u^0)\in A$ is a feasible solution for (MOP) and there exists $\lambda^0 \in R^k$, $\lambda^0 \geq 0$, $\beta^0 \in PS(I,R^m_+)$ such that (3.1)-(3.3) of theorem 3.1 are satisfied.

Further, assume that $(f,\beta^0(t)^T h)$ is weakly ρ-V-strictly pseudo univex type I at (x^0,u^0) with respect to functions ϕ_0, ϕ_1, b_0,b_1,η with $b_1(x,u,x^0,u^0)>0$ for all $(x,u)\in A$. Suppose that $\phi_1(0) \geqq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, provided $\lambda^{0T}\rho^1 + \beta^{0T}\rho^2 \geq 0$, then (x^0,u^0) is an efficient solution for (MOP).

Proof: If (x^0,u^0) is not an efficient solution for (MOP), then there exists $(x,u)\in A$ such that $\int_a^b f(t,x,\dot{x},u,\dot{u})dt \leq \int_a^b f(t,x^0,\dot{x}^0,u^0,\dot{u}^0)dt$.

From (3.2), we have $\int_a^b \beta^0(t)^T h(t,x^0,\dot{x}^0,u^0,\dot{u}^0)dt = 0$.

Using $\phi_1(0) \geqq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, we get

$$\phi_0\left[\int_a^b f(t,x,\dot{x},u,\dot{u})dt - \int_a^b f(t,x^0,\dot{x}^0,u^0,\dot{u}^0)dt\right] \leq 0,$$

$$-\phi_1\left[\int_a^b \beta^0(t)^T h(t,x^0,\dot{x}^0,u^0,\dot{u}^0)dt\right] \leqq 0.$$

Since $(f,\beta^0(t)^t h)$ is weakly ρ-V-strictly pseudo univex type I at (x^0,u^0) with respect to ϕ_0, ϕ_1, b_0,b_1,η,

$$b_0(x,u,x^0,u^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^T(f_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}f_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)) \end{array}\right]dt \tag{3.9}$$

$$< -\rho^1 d^2(x,u,x^0,u^0),$$

$$b_1(x,u,x^0,u^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^T(\beta^0(t)^T h_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}\beta^0(t)^T h_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)) \end{array}\right]dt \tag{3.10}$$

$$< -(\beta^{0T}\rho^2)d^2(x,u,x^0,u^0).$$

From (3.9) and (3.10), we have

$$\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^T(f_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}f_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)) \end{array}\right]dt \tag{3.11}$$

$$< -\rho^1 d^2(x,u,x^0,u^0),$$

$$\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^T(\beta^0(t)^T h_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}\beta^0(t)^T h_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)) \end{array}\right]dt \tag{3.12}$$

$$< -(\beta^{0T}\rho^2)d^2(x,u,x^0,u^0).$$

Since $\lambda^0 \geqq 0$, (3.11) gives

$$\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^T(\lambda^{0T}f_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) - \frac{d}{dt}\lambda^{0T}f_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0)) \end{array}\right]dt \tag{3.13}$$

$$\leqq -(\lambda^{0T}\rho^1)d^2(x,u,x^0,u^0).$$

Adding (3.12) and (3.13), we obtain

$$\int_a^b F\begin{bmatrix} t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0; \\ \eta(t,x,u,x^0,u^0)^T(\lambda^{0T}f_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) + \beta^0(t)^T h_x(t,x^0,u^0,\dot{x}^0,\dot{u}^0) \\ -\frac{d}{dt}(\lambda^{0T}f_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0) + \beta^0(t)^T h_{\dot{x}}(t,x^0,u^0,\dot{x}^0,\dot{u}^0))) \end{bmatrix} dt$$
$$< -(\lambda^{0T}\rho^1 + \beta^{0T}\rho^2)d^2(x,u,x^0,u^0),$$

which contradicts (3.1). Hence the result.

Mixed Type Duality

We divide the index set M of the constraint function of the problem (MOP) into two distinct subsets, namely J_1 and J_2 such that $J_1 \cup J_2 = M$, and let e be the vector of R^k whose components are all ones. We consider the following mixed type dual for (MOP):

$$\text{(XMOP) Maximize } \int_a^b [(f(t,z,\dot{z},w,\dot{w}) + \{\beta_{J_1}^T(t)h^{J_1}(t,z,\dot{z},w,\dot{w})\}e]dt,$$

subject to x(a) = α, x(b) = β,

$$\left[\lambda^T f_u(t,z,\dot{z},w,\dot{w}) + \beta(t)^T h_u(t,z,\dot{z},w,\dot{w})\right]$$
$$= \frac{d}{dt}\left[\lambda^T f_{\dot{u}}(t,z,\dot{z},w,\dot{w}) + \beta(t)^T h_{\dot{u}}(tz,\dot{z},w,\dot{w})\right], \tag{4.1}$$

$$\int_a^b \beta_{J_2}^T(t)h^{J_2}(t,z,\dot{z},w,\dot{w})dt \geqq 0, \tag{4.2}$$

$$\beta(t) \geq 0, \ t \in I, \tag{4.3}$$

$$\lambda \in R^k, \lambda \geqq 0, \lambda^T e = 1, e = (1,1,..,1) \in R^k. \tag{4.4}$$

Let B be the set of feasible solutions of (XMOP).

We note that we get a Mond-Weir [14] type dual for $J_1 = \varnothing$ and a Wolfe [25] type dual for $J_2 = \varnothing$ in (XMOP) respectively.

We prove various duality results for (MOP) and (XMOP) under generalized ρ-V-univexity type I conditions.

Theorem 4.1: (Weak Duality): Let $(x,u) \in A$ and $(z,w,\lambda,\beta(t)) \in B$. Let any of the following conditions holds:

a. $\lambda > 0$, $(f + \beta_{J_1}(t)^T h_{J_1} e, \beta_{J_2}(t)^T h_{J_2})$ is strongly ρ-V-pseudoquasi univex type I at (z,w) with respect to ϕ_0, ϕ_1, b_0, b_1, η with $b_1(x,u,z,w) > 0$ for all $(x,u) \in A$. Also $a \leq 0 \Rightarrow \phi_0(a) \leq 0$ and $a \geqq 0 \Rightarrow \phi_1(a) \geqq 0$, provided $(\lambda^T \rho^1 + \beta^T \rho^2) \geq 0$,

b. $(f + \beta_{J_1}(t)^T h_{J_1} e, \beta_{J_2}(t)^T h_{J_2})$ is weakly ρ-V-strictly pseudoquasi univex type I at (z,w) with respect to ϕ_0, ϕ_1, b_0, b_1, η with $b_1(x,u,z,w) > 0$ for all $(x,u) \in A$. Also $a \leq 0 \Rightarrow \phi_0(a) \leq 0$ and $a \geqq 0 \Rightarrow \phi_1(a) \geqq 0$, provided $(\lambda^T \rho^1 + \beta^T \rho^2) \geq 0$,

c. $(f + \beta_{J_1}(t)^T h_{J_1} e, \beta_{J_2}(t)^T h_{J_2})$ is weakly ρ-V-strictly pseudo univex type I at (z,w) with respect to ϕ_0, ϕ_1, b_0, b_1, η with $b_1(x,u,z,w)>0$ for all $(x,u) \in K$. Also $a \leq 0 \Rightarrow \phi_0(a) \leq 0$ and $a \geqq 0 \Rightarrow \phi_1(a) \geqq 0$, provided $(\lambda^T \rho^1 + \beta^T \rho^2) \geq 0$, then the following cannot hold:

$$\int_a^b f(t,x,\dot{x},u,\dot{u})dt \leq \int_a^b [f(t,z,\dot{z},w,\dot{w}) + \{\beta_{J_1}(t)^T h^{J_1}(t,z,\dot{z},w,\dot{w})\}e]dt.$$

Proof: Let (x,u) be feasible for (MOP) and $(z,w,\lambda,\beta(t))$ be feasible for (XMOP). Suppose that

$$\int_a^b f(t,x,\dot{x},u,\dot{u})dt \leq \int_a^b [f(t,z,\dot{z},w,\dot{w}) + \{\beta_{J_1}(t)^T h^{J_1}(t,z,\dot{z},w,\dot{w})\}e]dt.$$

Since (x,u) is feasible for (MOP) and $(z,w,\lambda,\beta(t))$ be feasible for (XMOP), we have

$$\int_a^b [f(t,x,\dot{x},u,\dot{u}) + \{\beta_{J_1}(t)^T h^{J_1}(t,x,\dot{x},u,\dot{u})\}e]dt \tag{4.5}$$
$$\leq \int_a^b [f(t,z,\dot{z},w,\dot{w}) + \{\beta_{J_1}(t)^T h^{J_1}(t,z,\dot{z},w,\dot{w})\}e]dt.$$

Using $a \geqq 0 \Rightarrow \phi_1(a) \geqq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$ with (3.1), we get

$$\phi_0\begin{bmatrix} \int_a^b [f(t,x,\dot{x},u,\dot{u}) + \{\beta_{J_1}(t)^T h^{J_1}(t,x,\dot{x},u,\dot{u})\}e]dt \\ - \int_a^b [f(t,z,\dot{z},w,\dot{w}) + \{\beta_{J_1}(t)^T h^{J_1}(t,z,\dot{z},w,\dot{w})\}e]dt \end{bmatrix} \leq 0,$$

$$-\phi_1\left[\int_a^b \beta_{J_2}(t)^T h^{J_2}(t,x,\dot{x},u,\dot{u})dt\right] \leqq 0.$$

Since $(f + \beta_{J_1}(t)^T h_{J_1} e, \beta_{J_2}(t)^T h_{J_2})$ is strongly ρ-V-pseudoquasi univex type I at (z,w) with respect to ϕ_0, ϕ_1, b_0, b_1, η

$$b_0(x,z,u,w)\int_a^b F\begin{bmatrix} t,x,\dot{x},u,\dot{u},z,\dot{z},w,\dot{w}; \\ \eta(t,x,u,z,w)^T(f_u(t,z,\dot{z},w,\dot{w}) + e\beta_{J_1}(t)^T h_{J_1 u}(t,z,\dot{z},w,\dot{w})) \\ -\frac{d}{dt}(f_{\dot{u}}(t,z,\dot{z},w,\dot{w}) + e\beta_{J_1}(t)^T h_{J_1 u}(t,z,\dot{z},w,\dot{w})) \end{bmatrix} dt$$

$$< -\rho^1 d^2(x,u,z,w),$$

$$b_1(x,z,u,w)\int_a^b F\begin{bmatrix} t,x,\dot{x},u,\dot{u},z,\dot{z},w,\dot{w};\ \eta(t,x,u,z,w)^T(\beta_{J_2}(t)^T h_{J_2 u}(t,,z,\dot{z},w,\dot{w}) \\ -\frac{d}{dt}(\beta_{J_2}(t)^T h_{J_2 \dot{u}}(t,z,\dot{z},w,\dot{w}))) \end{bmatrix} dt$$

$$\leqq -(\beta^T \rho^2) d^2(x,u,z,w).$$

Since $b_1(x,z,u,w) > 0$ and $\lambda^T > 0$, we get

$$b_0(x,z,u,w)\int_a^b F\begin{bmatrix} t,x,\dot{x},u,\dot{u},z,\dot{z},w,\dot{w}; \\ \eta(t,x,u,z,w)^T((\lambda^T f_u(t,z,\dot{z},w,\dot{w}) + e\beta_{J_1}(t)^T h_{J_1 u}(t,z,\dot{z},w,\dot{w})) \\ -\frac{d}{dt}(\lambda^T f_{\dot{u}}(t,z,\dot{z},w,\dot{w}) + e\beta_{J_1}(t)^T h_{J_1 u}(t,z,\dot{z},w,\dot{w})) \end{bmatrix} dt \tag{4.6}$$

$$< -(\lambda^T \rho^1) d^2(x,u,z,w),$$

$$\int_a^b F\begin{bmatrix} t,x,\dot{x},u,\dot{u},z,\dot{z},w,\dot{w};\ \eta(t,x,u,z,w)^T(\beta_{J_2}(t)^T h_{J_2 u}(t,z,\dot{z},w,\dot{w}) \\ -\frac{d}{dt}(\beta_{J_2}(t)^T h_{J_2 \dot{u}}(t,z,\dot{z},w,\dot{w}))) \end{bmatrix} dt \tag{4.7}$$

$$\leqq -(\beta^T \rho^2) d^2(x,u,z,w).$$

By $b_0(x,z,u,w) \geqq 0$, it follows that

$$b_0(x,z,u,w)\int_a^b F\begin{bmatrix} t,x,\dot{x},u,\dot{u},z,\dot{z},w,\dot{w};\ \eta(t,x,u,z,w)^T(\beta_{J_2}(t)^T h_{J_2 u}(t,z,\dot{z},w,\dot{w}) \\ -\frac{d}{dt}(\beta_{J_2}(t)^T h_{J_2 \dot{u}}(t,z,\dot{z},w,\dot{w}))) \end{bmatrix} dt \tag{4.8}$$

$$\leqq -(\beta^T \rho^2) d^2(x,u,z,w).$$

Adding (4.6) and (4,8), we obtain

$$b_0(x,z,u,w)\int_a^b F\begin{bmatrix} t,x,\dot{x},u,\dot{u},z,\dot{z},w,\dot{w}; \\ \eta(t,x,u,z,w)^T((\lambda^T f_u(t,z,\dot{z},w,\dot{w}) + \beta(t)^T h_u(t,z,\dot{z},w,\dot{w})) \\ -\frac{d}{dt}(\lambda^T f_{\dot{u}}(t,z,\dot{z},w,\dot{w}) + \beta(t)^T h_{\dot{u}}(t,z,\dot{z},w,\dot{w})) \end{bmatrix}dt$$
$$< -(\lambda^T\rho^1+\beta^T\rho^2)d^2(x,u,z,w),$$

which contradicts (4.1).

Now, by hypothesis (b) and from (4.2), (4.5), we get

$$\phi_0\begin{bmatrix} \int_a^b[f(t,x,\dot{x},u,\dot{u}) + \{\beta_{J_1}(t)^T h^{J_1}(t,x,\dot{x},u,\dot{u})\}e]dt \\ -\int_a^b[f(t,z,\dot{z},w,\dot{w}) + \{\beta_{J_1}(t)^T h^{J_1}(t,z,\dot{z},w,\dot{w})\}e]dt \end{bmatrix} \leq 0,$$
$$-\phi_1\left[\int_a^b \beta_{J_2}(t)^T h^{J_2}(t,x,\dot{x},u,\dot{u})dt\right] \leqq 0.$$

Since $(f + \beta_{J_1}(t)^T h_{J_1}e, \beta_{J_2}(t)^T h_{J_2})$ is weakly ρ-V-strictly pseudoquasi univex type I at (z,w) with respect to ϕ_0, ϕ_1, b_0, b_1, η

$$b_0(x,z,u,w)\int_a^b F\begin{bmatrix} t,x,\dot{x},u,\dot{u},z,\dot{z},w,\dot{w}; \\ \eta(t,x,u,z,w)^T((f_u(t,z,\dot{z},w,\dot{w}) + e\beta_{J_1}(t)^T h_{J_1u}(t,z,\dot{z},w,\dot{w})) \\ -\frac{d}{dt}(f_{\dot{u}}(t,z,\dot{z},w,\dot{w}) + e\beta_{J_1}(t)^T h_{J_1\dot{u}}(t,z,\dot{z},w,\dot{w})) \end{bmatrix}dt$$
$$< -\rho^1 d^2(x,u,z,w),$$

$$b_1(x,z,u,w\int_a^b F\begin{bmatrix} t,x,\dot{x},u,\dot{u},z,\dot{z},w,\dot{w};\ \eta(t,x,u,z,w)^T(\beta_{J_2}(t)^T h_{J_2u}(t,,z,\dot{z},w,\dot{w}) \\ -\frac{d}{dt}(\beta_{J_2}(t)^T h_{J_2\dot{u}}(t,z,\dot{z},w,\dot{w}))) \end{bmatrix}dt$$
$$\leqq -(\beta^T\rho^2)d^2(x,u,z,w).$$

Since $b_1(x,z,u,w) > 0$ and $\lambda^T \geq 0$, we get

$$b_0(x,z,u,w)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},z,\dot{z},w,\dot{w}; \\ \eta(t,x,u,z,w)^T((\lambda^T f_u(t,z,\dot{z},w,\dot{w}) + e\beta_{J_1}(t)^T h_{J_1 u}(t,z,\dot{z},w,\dot{w})) \\ -\frac{d}{dt}(\lambda^T f_{\dot{u}}(t,z,\dot{z},w,\dot{w}) + e\beta_{J_1}(t)^T h_{J_1\dot{u}}(t,z,\dot{z},w,\dot{w})) \end{array}\right]dt \tag{4.9}$$

$< -\rho^2 d^2(x,u,z,w),$

$$\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},z,\dot{z},w,\dot{w};\ \eta(t,x,u,z,w)^T(\beta_{J_2}(t)^T h_{J_2 u}(t,,z,\dot{z},w,\dot{w}) \\ -\frac{d}{dt}(\beta_{J_2}(t)^T h_{J_2\dot{u}}(t,z,\dot{z},w,\dot{w}))) \end{array}\right]dt \tag{4.10}$$

$\leqq -(\beta^T\rho^2)d^2(x,u,z,w).$

By $b_0(x,z,u,w) \geqq 0$, it follows that

$$b_0(x,z,u,w)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},z,\dot{z},w,\dot{w};\ \eta(t,x,u,z,w)^T(\beta_{J_2}(t)^T h_{J_2 u}(t,z,\dot{z},w,\dot{w}) \\ -\frac{d}{dt}(\beta_{J_2}(t)^T h_{J_2\dot{u}}(t,z,\dot{z},w,\dot{w}))) \end{array}\right]dt \tag{4.11}$$

$\leqq -(\beta^T\rho^2)d^2(x,u,z,w).$

Adding (4.9) and (4.11), we obtain

$$b_0(x,z,u,w)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},z,\dot{z},w,\dot{w};\eta(t,x,u,z,w)^T((\lambda^T f_u(t,z,\dot{z},w,\dot{w})+\beta(t)^T h_u(t,z,\dot{z},w,\dot{w})) \\ -\frac{d}{dt}(\lambda^t f_{\dot{u}}(t,z,\dot{z},w,\dot{w}) + \beta(t)^T h_{\dot{u}}(t,z,\dot{z},w,\dot{w}))) \end{array}\right]dt$$

$< -\left(\lambda^T\rho^1 + \beta^T\rho^2\right)d^2(x,u,z,w),$

which contradicts (4.1).

If (c) holds, then from (4.2) and (4.5), we get

$$\phi_0\left[\begin{array}{l} \int_a^b [f(t,x,\dot{x},u,\dot{u}) + \{\beta_{J_1}(t)^T h^{J_1}(t,x,\dot{x},u,\dot{u})\}e]dt \\ -\int_a^b [f(t,z,\dot{z},w,\dot{w}) + \{\beta_{J_1}(t)^T h^{J_1}(t,z,\dot{z},w,\dot{w})\}e]dt \end{array}\right] \leq 0,$$

$$-\phi_1\left[\int_a^b \beta_{J_2}(t)^T h^{J_2}(t,x,\dot{x},u,\dot{u})dt\right] \leqq 0.$$

Since $(f + \beta_{J_1}(t)^T h_{J_1}e,\ \beta_{J_2}(t)^T h_{J_2})$ is weakly ρ-V-strictly pseudo univex type I at (z,w) with respect to ϕ_0, ϕ_1, b_0, b_1, η

$$b_0(x,z,u,w)\int_a^b F\begin{bmatrix} t,x,\dot{x},u,\dot{u},z,\dot{z},w,\dot{w}; \\ \eta(t,x,u,z,w)^T((f_u(t,z,\dot{z},w,\dot{w}) + e\beta_{J_1}(t)^T h_{J_1 u}(t,z,\dot{z},w,\dot{w})) \\ -\frac{d}{dt}(f_{\dot{u}}(t,z,\dot{z},w,\dot{w}) + e\beta_{J_1}(t)^T h_{J_1\dot{u}}(t,z,\dot{z},w,\dot{w})) \end{bmatrix} dt \tag{4.12}$$

$$< -\rho^1 d^2(x,u,z,w),$$

$$b_1(x,z,u,w)\int_a^b F\begin{bmatrix} t,x,\dot{x},u,\dot{u},z,\dot{z},w,\dot{w};\ \eta(t,x,u,z,w)^T(\beta_{J_2}(t)^T h_{J_2 u}(t,z,\dot{z},w,\dot{w}) \\ -\frac{d}{dt}(\beta_{J_2}(t)^T h_{J_2\dot{u}}(t,z,\dot{z},w,\dot{w}))) \end{bmatrix} dt \tag{4.13}$$

$$< -(\beta^T\rho^2)d^2(x,u,z,w).$$

From (4.12) and (4.13), we get

$$\int_a^b F\begin{bmatrix} t,x,\dot{x},u,\dot{u},z,\dot{z},w,\dot{w}; \\ \eta(t,x,u,z,w)^T((f_u(t,z,\dot{z},w,\dot{w}) + e\beta_{J_1}(t)^T h_{J_1 u}(t,z,\dot{z},w,\dot{w})) \\ -\frac{d}{dt}(f_{\dot{u}}(t,z,\dot{z},w,\dot{w}) + e\beta_{J_1}(t)^T h_{J_1\dot{u}}(t,z,\dot{z},w,\dot{w})) \end{bmatrix} dt \tag{4.14}$$

$$< -\rho^1 d^2(x,u,z,w),$$

$$\int_a^b F\begin{bmatrix} t,x,\dot{x},u,\dot{u},z,\dot{z},w,\dot{w};\ \eta(t,x,u,z,w)^T(\beta_{J_2}(t)^T h_{J_2 u}(t,z,\dot{z},w,\dot{w}) \\ -\frac{d}{dt}(\beta_{J_2}(t)^T h_{J_2\dot{u}}(t,z,\dot{z},w,\dot{w}))) \end{bmatrix} dt \tag{4.15}$$

$$< -(\beta^T\rho^2)d^2(x,u,z,w).$$

Because $\lambda^T \geq 0$, (4.14) gives

$$\int_a^b F\begin{bmatrix} t,x,\dot{x},u,\dot{u},z,\dot{z},w,\dot{w}; \\ \eta(t,x,u,z,w)^T((\lambda^T f_u(t,z,\dot{z},w,\dot{w}) + \beta_{J_1}(t)^T h_{J_1 u}(t,z,\dot{z},w,\dot{w})) \\ -\frac{d}{dt}(\lambda^T f_{\dot{u}}(t,z,\dot{z},w,\dot{w}) + \beta_{J_1}(t)^T h_{J_1\dot{u}}(t,z,\dot{z},w,\dot{w})) \end{bmatrix} dt \tag{4.16}$$

$$< -(\lambda^T\rho^1)d^2(x,u,z,w),$$

Adding (4.15) and (4.16), we get

$$\int_a^b F\left[\begin{array}{l} t,x,\dot{x},u,\dot{u},z,\dot{z},w,\dot{w};\ \eta(t,x,u,z,w)^T((\lambda^T f_u(t,z,\dot{z},w,\dot{w}) + \beta(t)^T h_u(t,z,\dot{z},w,\dot{w})) \\ -\dfrac{d}{dt}(\lambda^T f_{\dot{u}}(t,z,\dot{z},w,\dot{w}) + \beta(t)^T h_{\dot{u}}(t,z,\dot{z},w,\dot{w})) \end{array}\right]dt$$
$$< -(\lambda^T\rho^1+\beta^T\rho^2)d^2(x,u,z,w),$$

which contradicts (4.1).

Corollary 4.1: (See [1] Let $(z^0,w^0,\lambda^0,\beta^0(t))$ be a feasible solution for (XMOP). Assume that $\beta^0_{J_1}(t)^T h_{J_1}(t,z^0,\dot{z}^0,w^0,\dot{w}^0) = 0$ and assume that (z^0,w^0) is a feasible for (MOP). If the weak duality theorem 4.1 holds between (MOP) and (XMOP), then (z^0,w^0) is an efficient solution for (MOP) and $(z^0,w^0,\lambda^0,\beta^0(t))$ is an efficient solution for (XMOP).

Necessary optimality conditions for the existence of an external solution for the single objective variational problem subject to both equality and inequality constraints were given by Valentine [22]. Invoking Valentine's [22] results, Hanson and Mond [8] obtained corresponding necessary optimality conditions. Using the relationship between the efficient solution of the problem (MOP) and the optimal solution of the associated scalar control problem, the necessary optimality conditions were derived for the multiobjective variational problems; details can be found in [6]. Fritz John necessary optimality conditions derived in the form of (3.1)-(3.3) of theorem 3.1 with $\lambda^0 \geqq 0$, lead to Kuhn-Tucker type necessary optimality conditions under additional constraint qualifications.

Theorem 4.2: (Strong Duality, [1]): Let (x^0,u^0) be feasible solution for (MOP) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exists $\lambda^0 \in R^k$, $\lambda^0 \geq 0$, $\lambda^{0T}e = 1$, $\beta^0 \in PS(I,R^m_+)$ such that $(x^0,u^0,\lambda^0,\beta^0(t))$ is feasible for (XMOP) with $\beta^0_{J_1}(t)^T h_{J_1}(t,z^0,\dot{z}^0,w^0,\dot{w}^0) = 0$.

If also the weak duality theorem 3.1 holds between (MOP) and (XMOP), then $(x^0,u^0,\lambda^0,\beta^0(t))$ is an efficient solution for (XMOP).

Proof: Since (x^0,u^0) is an efficient solution for (MOP) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exists $\lambda^0 \in R^k$, $\lambda^0 \geq 0$, $\lambda^{0T}e = 1$, $\beta^0 \in PS(I,R^m_+)$ such that (3.1)-(3.3) of theorem 3.1 hold. Moreover, $(x^0,u^0) \in A$, hence the feasibility of $(x^0,u^0,\lambda^0,\beta^0(t))$ for (XMOP) follows.

Also because weak duality holds between (MOP) and (XMOP), $(x^0,u^0,\lambda^0,\beta^0(t))$ is an efficient solution for (XMOP).

If $(x^0,u^0,\lambda^0,\beta^0(t))$ is not an efficient solution for (XMOP), then proceeding along the lines similar to those in Corollary 4.1 in [1], we a get a contradiction to weak duality.

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