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# On the Solution of Integro-Differential Equation Systems by Using ELzaki Transform.

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**Abstract**

In this work a new integral transform, namely ELzaki transform was applied to solve linear systems of Integra-differential equations with constant coefficients.

**Keywords:** ELzaki transform -Systems -Integro-differential equation.

## Introduction

Many problems of physical interest are described by differential and integral equations with appropriate or boundary conditions. These problems are usually formulated as initial value problem, boundary value problems, or initial – boundary value problem that seem to be mathematically more vigorous and physically realistic in applied and engineering sciences. ELzaki transform method is very effective for Solution of the response of differential and integral equations and a linear system of differential and integral equations.

The technique that we used is ELzaki transform method which is based on Fourier transform. It introduced by Tarig ELzaki (2010)

In this study, ELzaki transform is applied to integral and integrao-differential equations system which the solution of these equations have a major role in the fields of science and engineering. When a physical system is modeled under the differential sense, if finally gives a differential equation, an integral equation or an integro-differential equation systems.

Recently, Tarig ELzaki introduced a new transform and named as ELzaki transform which is defined by:

$$\mathrm{E}\left[f(t),v\right]=T(v)=v\int_0^\infty e^{-\frac{t}{v}}f(t)\,dt \quad , \quad v\in\left(-k_1,k_2\right)$$

Or for a function $f\ (t)$ which is of exponential order,

$$\left|f\ (t)\right| < \begin{cases} Me^{-t/k_1} & , \quad t \le 0 \\ Me^{t/k_2} & , \quad t \ge 0 \end{cases}$$

ELzaki transform, henceforth designated by the operator $\mathrm{E}[.]$, is defined by the integral equation.

$$\mathrm{E}\left[f(t)\right] = T(v) = v^2\int_0^\infty f(vt)\ e^{-t}dt \ ,\ -k_1 \le v \le k_2$$

Where $M$ is a real finite number and $k_1, k_2$ can be finite or infinite.

**Theorem (1-1):**
Let $T\ (v)$ is ELzaki transform of $f(t)$

$$\left[\mathrm{E}\left(f(t)\right) = T(v)\right] \quad and \quad g(t) = \begin{cases} f(t-\tau), & t \ge \tau \\ 0 \quad , & t < \tau \end{cases}$$

Then:

$$\mathrm{E}\left[g(t)\right] = e^{\frac{-\tau}{v}}\ T(v)$$

**Proof:**

$$\mathrm{E}\left[g\ (t)\right] = \int_\tau^\infty v\ e^{-t/v} f\ (t-\tau)dt \text{ Let } t = \lambda + \tau \text{ we find that:}$$

$$\int_0^\infty v e^{-(\frac{\lambda+\tau}{v})} f(\lambda)d\lambda = e^{-\tau/v}\int_0^\infty v\ e^{-\lambda/v}\ f(\lambda)d\lambda = e^{-\frac{\tau}{v}}T(v)$$

Which is the desired result

ELzaki transform can certainly treat all problems that are usually treated by the well- known and extensively used Laplace transform.

Indeed as the next theorem shows ELzaki transform is closely connected with the Laplace transform $F\ (s)$.

**Theorem (1-2):**
Let $f(t) \in A = \left\{ f(t) \middle| \exists M, k_1, k_2 > 0, such \quad that \middle|, \left|f(t)\right| < Me^{|t|/k_i} \ ,\quad if\ t \in (-1)^j \times [0, \infty) \right\}$

With Laplace transform $F(s)$, Then ELzaki transform $T(v)$ of $f(t)$ is given by

$$T(v) = vF\left(\frac{1}{v}\right) \tag{1}$$

**Proof:**

Let: $f(t) \in A$ . Then for $-k_1 < v < k_2$ $T(v) = v^2 \int_0^\infty e^{-t} f(vt)\,dt$ Let $w = vt$ then we have:

$$T(v) = v^2 \int_0^\infty e^{-\frac{w}{v}} f(w) \frac{dw}{v} = v \int_0^\infty e^{-\frac{w}{v}} f(w)\,dw = vF\left(\frac{1}{v}\right).$$

Also we have that $T(1) = F(1)$ so that both the ELzaki and Laplace transforms must coincide at $v = s = 1$.

In fact the connection of the ELzaki transform with the Laplace transform goes much deeper, therefore the rules of $F$ and $T$ in (1) can be interchanged by the following corollary.

**Corollary (1-3):**
Let $f(t)$ having $F$ and $T$ for Laplace and ELzaki transforms respectively, then:

$$F(s) = sT\left(\frac{1}{s}\right) \tag{2}$$

**Proof:**

This relation can be obtained from (1) by taking $v = \frac{1}{s}$

Equations (1) and (2) are the duality relation governing these two transforms and may serve as a mean to get one from the other when needed.

## ELzaki Transform of Derivatives and Integrals

Being restatement of the relation (1) will serve as our working definition, since the Laplace transform of $\sin t$ is $\frac{1}{1+s^2}$ then view of (1), its. ELzaki transform is $\mathrm{E}[\sin t] = \frac{v^3}{1+v^2}$ this exemplifies the duality between these two transforms.

**Theorem (2-1):**
Let $F'(s)$ and $T'(v)$ be Laplace and ELzaki transforms of the derivative of $f(t)$. Then:

$$(i)\ T'(v) = \frac{T(v)}{v} - v\,f(0) \tag{3}$$

$$(ii)\ T^{(n)}(v)=\frac{T\ (v)}{v^{n}}-\sum_{k=0}^{n-1} v^{2-n+k}\ f^{(k)}(0) \qquad , \quad n\geq 1 \tag{4}$$

Where $T^{n}(v)$ *and* $F^{n}(s)$ are ELzaki and Laplace transforms of the nth derivative $f^{(n)}(t)$ of the function $f\ (t)$.

**Proof:**
(i) Since the Laplace transform of the derivatives of $f(t)$ is $F'(s)=sF(s)-f(0)$

**Then:**

$$T'(v)=vF'\left(\frac{1}{v}\right)=v\left[\frac{1}{v}F\left(\frac{1}{v}\right)-f(0)\right]=F\left(\frac{1}{v}\right)-v\ f(0)=\ \frac{T(v)}{v}-v\ f(0)$$

(ii) By definition, the Laplace transform for $f^{(n)}(t)$ is given by:

$$F^{(n)}(s)=S^{n}F(s)-\sum_{k=0}^{n-1}S^{n-(k+1)}f^{(k)}(0)$$

**Therefore:**

$$F\left(\frac{1}{v}\right)=\frac{F\left(\frac{1}{v}\right)}{v^{n}}\ -\ \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{v^{n-(k+1)}}$$

Now, since,

$$T^{(k)}(v)=v\ F^{(k)}\left(\frac{1}{v}\right)\quad for\quad 0\leq k\leq m\ ,\ we\ \ have\ \ T^{(n)}(v)=\frac{T(v)}{v^{n}}-\sum_{k=0}^{n-1}v^{2-n+k}f^{(k)}(0)$$

**Theorem (2-2)**
Let $T'(v)$ *and* $F'(s)$ denote ELzaki and Laplace transforms of the definite integral of $f\ (t)$.

$$h(t)=\int_{0}^{t}f(\tau)\ d\tau.\quad Then\qquad T'(v)=E\left[h(t)\right]=vT(v)$$

**Proof:**

By the definition of Laplace transform $F'(s)=L\big(h(t)\big)=\dfrac{F(s)}{s}$ Hence,

$$T'(v)=vF'\left(\frac{1}{v}\right)=v\left[vF\left(\frac{1}{v}\right)\right]=v^{2}F\left(\frac{1}{v}\right)=vT(v)$$

**Theorem (2-3) (shift):**
Let $f\,(t) \in A$ with ELzaki transform $T\,(v)$.Then:

$$E\left[e^{at} f(t)\right] = \frac{1}{1-av}\ \ T\left[\frac{v}{1-av}\right]$$

**Proof:**
From definition of ELzaki transform we have:

$$E\left[e^{at} f(t)\right] = v^2 \int_0^{\infty} f(vt) e^{-(1-av)t} dt \text{ Let } w = (1-av)t \Rightarrow dw = (1-av)dt,$$

**Then:**

$$\frac{v^2}{1-av}\int_0^{\infty} f\left[\frac{wv}{1-av}\right] e^{-w} dw = \frac{1}{1-av}\ \ T\left[\frac{v}{1-av}\right]$$

**Theorem (2-4) (convolution):**
Let $f(t)$ *and* $g(t)$ be defined in $A$ having Laplace transforms $F(s)$ and $G(s)$ and ELzaki transforms $M(v)$ and $N(v)$. Then ELzaki transform of the Convolution of $f$ and $g$.

$$(f*g)(t) = \int_0^{\infty} f(t) g(t-\tau) d\tau \text{ Is given by: } E\left[(f*g)(t)\right] = \frac{1}{v} M(v) N(v)$$

**Proof:**
The Laplace transform of $(f\ *g)$ is given by: $L\left[(f*g)\right] = F(s)G(s)$

By the duality relation (1) we have:

$$E\left[(f*g)(t)\right] = v\,L\left[(f*g)(t)\right],$$

And since

$$M(v) = vF\left(\frac{1}{v}\right),\ N(v) = vG\left(\frac{1}{v}\right)\ \ \textit{Then}\ \ E\left[(f*g)(t)\right] = v\left[F\left(\frac{1}{v}\right).G\left(\frac{1}{v}\right)\right]$$

$$= v\left[\frac{M(v)}{v}\ .\ \frac{N(v)}{v}\right] = \frac{1}{v} M(v) N(v)$$

## Solution System of Integro-Differential Equation

Let us consider the general first order system of Integro-differential equation.

$$\begin{cases} y_1' = f(t) + \int_o^t [y_1(x) + y_2(x)]\,dx \\ y_2' = g(t) + \int_0^t [y_2(x) - y_1(x)]\,dx \end{cases} \tag{5}$$

With the initial conditions

$$y_1(0) = \alpha \quad , \quad y_2(0) = \beta \tag{6}$$

By using ELzaki transform into eq (5) we have,

$$\begin{cases} \dfrac{\bar{y}_1}{v} - vy_1(0) = \bar{f}(v) + v\bar{y}_1 + v\bar{y}_2 \\ \dfrac{\bar{y}_2}{v} - vy_2(0) = \bar{g}(v) + v\bar{y}_2 - v\bar{y}_1 \end{cases} \tag{7}$$

Where $\bar{y}_1$ and $\bar{y}_2$ are ELzaki transform of $y_1$ and $y_2$ respectively.
Substituting Eq(6) into Eq(7) we get

$$\begin{cases} (1 - v^2)\,\bar{y}_1 = \alpha v^2 + v\,\bar{f}(v) + v^2 \bar{y}_2 \\ v^2 \bar{y}_1 = \beta v^2 + v\bar{g}(v) - (1 - v^2)\,\bar{y}_2 \end{cases}$$

Solve these equations to find $\bar{y}_1$ and $\bar{y}_2$

$$(2v^4 - 2v^2 + 1)\,\bar{y}_1 = \alpha v^2 + (\beta - \alpha)v^4 + v^2(1 - v^2)\bar{f}(v) + v^3 \bar{g}(v)$$

Or

$$\bar{y}_1 = \left[ \frac{\alpha v^2 + (\beta - \alpha)v^4 + v(1 - v^2)\,\bar{f}(v) + v^3 \bar{g}}{2v^4 - 2v^2 + 1} \right] = F(v)$$

And $y_1(t) = \mathrm{E}^{-1}[F(v)] = G(t)$
Substituting $y_1(t)$ into eq (5) to find $y_2(t)$.

**Example (1):**
Consider the following system:

$$\begin{cases} y_1' = t + \int_0^t \left[ y_1(x) + y_2(x) \right] dx \\ y_2' = -\frac{1}{12} t^4 - 2t + \int_0^t \left[ (t-x) . y_1(x) \right] dx \end{cases} \tag{8}$$

With the initial conditions

$$y_1(0) = 0 \quad , \quad y_2(0) = 1 \tag{9}$$

By using ELzaki transform into Eq (8) yields

$$\begin{cases} \frac{\bar{y}_1}{v} - v y_1(0) = v^3 + v\bar{y}_1 + v\bar{y}_2 \\ \frac{\bar{y}_2}{v} - v y_2(0) = -2v^6 - 2v^3 + \frac{1}{v}\left[ v^3 \bar{y}_1 \right] \end{cases} \tag{10}$$

Substituting Eq(9) into Eq (10) we get,

$$\begin{cases} \left(1 - v^2\right) \bar{y}_1 = v^4 + v^2 \bar{y}_2 \\ v^3 \bar{y}_1 = \bar{y}_2 + 2v^6 + 2v^4 - v^2 \end{cases}$$

Solve these algebraic equations we find that. $\bar{y}_1 = 2v^4$ And $y_1(t) = t^2$

Form the first equation of (8) we have $y_1' = t + \int_0^t \left[ y_1 + y_2 \right] dx$ or $\int_0^t y_2 dx = t - \frac{1}{3} t^3$

Applying ELzaki transform to the last equation, we get:

$v\bar{y}_2 = v^3 - 2v^5$ Or $\bar{y}_2 = v^2 - 2v^4$ And $y_2(t) = 1 - t^2$

**Example (2)**

Consider the following system.

$$\begin{cases} y_1'' = -1 - y_1 + \cos t + \int_0^t y_2(x) dx \\ y_2'' = -y_2 + \sin t - \int_0^t y_1(x) dx \end{cases} \tag{11}$$

With the initial conditions

$$y_1(0)=1 \quad , \quad y'_1(0)=0$$
$$y_2(0)=0 \quad , \quad y'_2(0)=1 \qquad (12)$$

**Solution:**
Applying ELzaki transform to Eq (11) we get:

$$\frac{\bar{y}_1}{v^2}-y_1(0)-vy_1'(0)=-v^2-\bar{y}_1+\frac{v^2}{1+v^2}+v\bar{y}_2$$
$$\frac{\bar{y}_2}{v^2}-y_2(0)-vy_2'(0)=-\bar{y}_2+\frac{v^3}{1+v^2}+v\bar{y}_1 \qquad (13)$$

Substituting Eq(12) into Eq(13) we have,

$$\left(1+v^2\right)\bar{y}_1=v^2-v^4+\frac{v^4}{1+v^2}+v^3\bar{y}_2$$
$$-v^3\bar{y}_1=v^3+\frac{v^5}{1+v^2}-\left(1+v^2\right)\bar{y}_2$$

The solution of these equations is,

$$\left(v^6+v^4+2v^2+1\right)\bar{y}_1= v^2+\frac{v^4}{1+v^2}+\frac{v^6}{1+v^2}+\frac{v^8}{1+v^2}$$
$$\bar{y}_1=\frac{v^2}{1+v^2} \text{ And } y_1(t)=\cos t$$

Substituting $y_1(t)$ into eq (11) we get: $\int_0^t y_2(x)dx=1-\cos t$

Take ELzaki transform of two side of this equation, we have,

$$v\bar{y}_2=v^2-\frac{v^2}{1+v^2} \text{ Or } \bar{y}_2=\frac{v^3}{1+v^2} \text{ and } y_2(t)=\sin t$$

**Example (3):**
Consider the following linear volterra type Integro-differential equation system.

$$\begin{cases} y_1'=1+t+t^2-y_2(t)-\int_0^t\left[y_1(x)+y_2(x)\right]dx \\ y_2'=-1-t+y_1(t)-\int_0^t\left[y_1(x)-y_2(x)\right]dx \end{cases} \qquad (14)$$

With the initial conditions.

$$y_1(0)=1 \quad , \quad y_2(0)=-1 \tag{15}$$

**Solution:**

By taking ELzaki transform of Eq (14) and making use of the conditions (15) we have:

$$\left(1+v^2\right)\bar{y}_1 = v^2+v^3+v^4+2v^5-v(1+v)\bar{y}_2$$
$$v(1-v)\bar{y}_2 = v^2+v^3+v^4+\left(1-v^2\right)\bar{y}_2$$

Solve these equations to find.

$$(1-v)\left(1+2v^2\right)\bar{y}_1 = -2v^6+2v^5+v^4+v^3-v^2$$

$$\bar{y}_1 = v^2\left[\frac{\left(-2v^4+2v^3-v^2+v\right)+\left(2v^2+1\right)}{(1-v)\left(1+2v^2\right)}\right]$$

$$\bar{y}_1 = v^2\left[v+\frac{1}{1-v}\right] = v^3+\frac{v^2}{1-v}\text{, then } y_1 = \mathrm{E}^{-1}\left[v^3+\frac{v^2}{1-v}\right] = t+e^t$$

Where that $\mathrm{E}^{-1}$ is the inverse ELzaki transform.

Substituting $y_1$ into equation (14) we get, $\bar{y}_2 = -\frac{1}{2}t^2+\int_0^t y_2(x)\,dx$

Applying ELzaki transform to this equation, we get.

$$\frac{\bar{y}_2}{v}+v = -v^4+v\bar{y}_2\text{, or } \left(1-v^2\right)\bar{y}_2 = -v^5-v^2$$

Or

$$\bar{y}_2 = -v^2\left[\frac{v^3+1}{1-v^2}\right] = v^2\left(v-\frac{1}{1-v}\right)$$

Then:

$$y_2 = \mathrm{E}^{-1}\left[v^3-\frac{v^2}{1-v}\right] = t-e^t$$

## Conclusion

In this paper, ELzaki transform method for the solution of volterra integral and Integro-differential equation systems is successfully expanded. In the first example, the general system of the first order Integro-differential equation and in the last three examples, Integro-differential equation systems are considered. In observed that ELzaki transform method is robust and is applicable to various types of Integro-differential and integral equation systems.

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