Global Journal of Mathematical Sciences: Theory and Practical.
Volume 3, Number 1 (2011), pp. 21-26

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Random Fixed Point for Sequence of Operators

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Keywords: Polish Space, Random multivalued operator, Random fixed point

Introduction

Random fixed point theorem for contraction mappings in polish spaces and random fixed point theorems are of fundamental importance in probabilistic functional analysis. Their study was initiated by the Prague school of Probabilistics with work of Spacek[15] and Hans[5,6]. For example survey are refer to Bharucha-Ried[4], Itoh[8] proved several random fixed point theorems and gave their applications to random differential equations in Banach spaces. Random coincidence point theorems and random fixed point theorem are stochastic generalization of classical coincidence point theorems and classical fixed point theorems. Sehgal and Singh[14], papageorgiou[12], Rhoades Sessa Khan[13] and Lin[11] have proved differential stochastic version of well known Schauder's fixed point theorem. Recently, Beg and Shahzad[3] studied the structure of common random fixed points and random coincidence points of a pair of compatible random operators.

The result of Hardy and Rogers[7] further extended by Wong[16], showing that two self mappings of S and T on a complete metric space satisfying a contractive type condition have a common fixed point. Recently, Beg and Azam[1] further extended it to the case of a pair of multivalued mappings satisfying a more general contractive type condition. In this section we gave a further generalized result of Beg and Shahzad[3] by using fractional inequality

Preliminaries

Let (X, d) be a polish space that is a saparable complete metric space and $(\Omega, \sum)$ be a measurable space with $\sum$ a sigma algebra of subsets of Ω. Let 2^X be the family of all non-empty subsets of *X* and *C*(X) the family of all nonempty closed subsets of *X*. A mapping $T : \Omega \rightarrow 2^X$ is called measurable if, for each open subset *U* of *X*,

$T^{-1}(U) \in \Sigma$, *where* $T^{-1}(U) = \{w \in \Omega : T(w) \cap U \neq \phi\}$.

A mapping $\xi : \Omega \to X$ is called a measurable selector of a measurable mapping $T : \Omega \to 2^X$ if ξ is measurable and $\xi(w) \in T(w)$ for each $w \in \Omega$. A mapping $f : \Omega \times X \to X$ is said to be a random operator if, for each fixed $x \in X$, $f(., x) : \Omega \to X$ is measurable. A measurable mapping $\xi : \Omega \to X$ is a random fixed point of a random multivalued operator $T : \Omega \times X \to C(X)$ ($f : \Omega \times X \to X$) if $\xi(w) \in T(w, \xi(w))$ [$\xi(w) = f(w, \xi(w))$] for each $w \in \Omega$. Let $T : \Omega \times X \to C(X)$ be a random operator and $\{\xi_n\}$ a sequence of measurable mappings $\xi_n : \Omega \to X$. The sequence $\{\xi_n\}$ is said to be asymptotically T-regular if $d(\xi_n(w), T(w, \xi_n(w))) \to 0$.

H represents the hausdroff metric on CB(X) induced by the metric d.

Main Result

Theorem: Let X be Polish Space. Let T_i, Sj: $\Omega \times X \to C(X)$ be sequence of continuous random multivalued operators for i,j =1,2…n….∞. If there exists measurable mappings a,b,c,d,e: $\Omega \to (0,1)$, such that

$$H(S_i(w,x),T_j(w,y)) \le a(w)d(x,y)+b(w)[d(x,S_i(w,x))+d(y,T_j(w,y))]+\frac{c(w)}{2}[d(x,T_j(w,y))+d(y,S_i(w,x))]$$

$$+d(w)[\frac{d(x,y)+d(y,S_i(w,x))+d(x,S_i(w,x))}{1+d(x,y)d(y,S_i(w,x))\,d(x,S_i(w,x))}]+\frac{e(w)}{2}[\frac{d(y,S_i(w,x))+\mathrm{d}(y,T_j(w,y))+\mathrm{d}(x,T_j(w,y))}{1+d(y,S_i(w,x))+\mathrm{d}(y,T_j(w,y))+\mathrm{d}(x,T_j(w,y))}]$$

$$\textit{for each}\; x, y \in X, w \in \Omega \;; i, j = 1,2.....n....\infty$$

$$\textit{and}\;\; a,b,c,d,e \in R^{+} \;\textit{with}\; 2[a(w)+c(w)]+4[b(w)+d(w)]+3e(w) < 2,$$

then there exists a common fixed point of S and T Proof : Let ξ_0: $\Omega \to X$ be an arbitrary measurable mapping and choose a measurable mapping

$\xi_1 : \Omega \to X$ *such that* $\xi_1(w) \in S_i(w, \xi_0(w))$ *for each* $w \in \Omega$

Therefore each $w \in \Omega$

$$H(S_i(w,\xi_0(w)),T_j(w,\xi_1(w))) \le a(w)d(\xi_0(w),\xi_1(w))+b(w)[d(\xi_0(w),S_i(w,\xi_0(w)))+d(\xi_1(w),T_j(w,\xi_1(w)))]$$

$$+\frac{c(w)}{2}[d(\xi_0(w),T_j(w,\xi_1(w)))+d(\xi_1(w),S_i(w,\xi_0(w)))]$$

$$+d(w)[\frac{d(\xi_0(w),\xi_1(w))+d(\xi_1(w),S_i(w,\xi_0(w)))+d(\xi_0(w),S_i(w,\xi_0(w)))}{1+d(\xi_0(w),\xi_1(w))d(\xi_1(w),S_i(w,\xi_0(w)))d(\xi_0(w),S_i(w,\xi_0(w)))}]$$

$$+\frac{e(w)}{2}[\frac{d(\xi_1(w),S_i(w,\xi_0(w)))+d(\xi_1(w),T_j(w,\xi_1(w)))+d(\xi_0(w),T_j(w,\xi_1(w)))}{1+d(\xi_1(w),S_i(w,\xi_0(w)))d(\xi_1(w),T_j(w,\xi_1(w)))d(\xi_0(w),T_j(w,\xi_1(w)))}]; i,j=1,2..n..\infty$$

It further implies, than there exists a measurable mapping

$\xi_2 : \Omega \to X$ *such that* $\xi_2(w) \in T_j(w, \xi_1(w))$ *for each* $w \in \Omega$ *and*

$$d(\xi_1(w), \xi_2(w)) = H(S_i(w, \xi_0(w)), T_j(w, \xi_1(w))$$

$$d(\xi_1(w), \xi_2(w)) \le a(w) d(\xi_0(w), \xi_1(w)) + b(w)[d(\xi_0(w), S_i(w, \xi_0(w))) + d(\xi_1(w), T_j(w, \xi_1(w)))]$$

$$+ \frac{c(w)}{2}[d(\xi_0(w), T_j(w, \xi_1(w))) + d(\xi_1(w), S_i(w, \xi_0(w)))]$$

$$+ d(w)[\frac{d(\xi_0(w), \xi_1(w)) + d(\xi_1(w), S_i(w, \xi_0(w))) + d(\xi_0(w), S_i(w, \xi_0(w)))}{1 + d(\xi_0(w), \xi_1(w)) d(\xi_1(w), S_i(w, \xi_0(w))) d(\xi_0(w), S_i(w, \xi_0(w)))}]$$

$$+ \frac{e(w)}{2}[\frac{d(\xi_1(w), S_i(w, \xi_0(w))) + d(\xi_1(w), T_j(w, \xi_1(w))) + d(\xi_0(w), T_j(w, \xi_1(w)))}{1 + d(\xi_1(w), S_i(w, \xi_0(w))) d(\xi_1(w), T_j(w, \xi_1(w))) d(\xi_0(w), T_j(w, \xi_1(w)))}]; i, j = 1, 2, \ldots n. \infty$$

$\in$

$$\le a(w) d(\xi_0(w), \xi_1(w)) + b(w)[d(\xi_0(w), \xi_1(w)) + d(\xi_1(w), \xi_2(w))]$$

$$+ \frac{c(w)}{2}[d(\xi_0(w), \xi_2(w)) + d(\xi_1(w), \xi_1(w))]$$

$$+ d(w)[\frac{d(\xi_0(w), \xi_1(w)) + d(\xi_1(w), \xi_1(w)) + d(\xi_0(w), \xi_1(w))}{1 + d(\xi_0(w), \xi_1(w)) d(\xi_1(w), \xi_1(w)) d(\xi_0(w), \xi_1(w))}]$$

$$+ \frac{e(w)}{2}[\frac{d(\xi_1(w), \xi_1(w)) + d(\xi_1(w), \xi_2(w)) + d(\xi_0(w), \xi_2(w))}{1 + d(\xi_1(w), \xi_1(w)) d(\xi_1(w), \xi_2(w)) d(\xi_0(w), \xi_2(w))}]$$

$$= a(w) d(\xi_0(w), \xi_1(w)) + b(w)\ [d(\xi_0(w), \xi_1(w)) + d(\xi_1(w), \xi_2(w))]$$

$$+ \frac{c(w)}{2} d(\xi_0(w), \xi_2(w)) + 2d(w) d(\xi_0(w), \xi_1(w))$$

$$+ \frac{e(w)}{2}[d(\xi_1(w), \xi_2(w)) + d(\xi_0(w), \xi_2(w))]$$

$$d(\xi_1(w), \xi_2(w)) \le k\ d(\xi_0(w), \xi_1(w))$$

$$\textit{where } k = \frac{a(w) + b(w) + c(w)/2 + 2d(w) + e(w)/2}{1 - b(w) - c(w)/2 - e(w)} < 1 \textit{ because } 2[a(w) + c(w)] + 4[b(w) + d(w)] + 3e(w) < 2$$

By above lemma in the same manner there exists a measurable mapping

$\xi_3 : \Omega \to X$ *such that* $\xi_3(w) \in S_i(w, \xi_2(w))$ *for each* $w \in \Omega$ *and*

$$d(\xi_2(w), \xi_3(w)) = H(T_j(w, \xi_1(w)), S_i(w, \xi_2(w))$$

$$\begin{aligned}
d(\xi_2(w), \xi_3(w)) \le\; & a(w)\, d(\xi_1(w), \xi_2(w)) + b(w)[\, d(\xi_1(w), T_j(w, \xi_1(w))) + d(\xi_2(w), S_i(w, \xi_2(w)))] \\
& + \frac{c(w)}{2}[d(\xi_1(w), S_i(w, \xi_2(w))) + d(\xi_2(w), T_j(w, \xi_1(w)))] \\
& + d(w)[\frac{d(\xi_1(w), \xi_2(w)) + d(\xi_2(w), T_j(w, \xi_1(w))) + d(\xi_1(w), T_j(w, \xi_1(w)))}{1 + d(\xi_1(w), \xi_2(w)) d(\xi_2(w), T(w, \xi_1(w))) d(\xi_1(w), T(w, \xi_1(w)))}] \\
& + \frac{e(w)}{2}[\frac{d(\xi_2(w), T_j(w, \xi_1(w))) + d(\xi_2(w), S_i(w, \xi_2(w))) + d(\xi_1(w), S_i(w, \xi_2(w)))}{1 + d(\xi_2(w), T_j(w, \xi_1(w))) d(\xi_2(w), S_i(w, \xi_2(w))) d(\xi_1(w), S_i(w, \xi_2(w)))}] \\
\le\; & a(w)\, d(\xi_1(w), \xi_2(w)) + b(w)[\, d(\xi_1(w), \xi_2(w)) + d(\xi_2(w), \xi_3(w))] \\
& + \frac{c(w)}{2}[d(\xi_1(w), \xi_3(w)) + d(\xi_2(w), \xi_2(w))] \\
& + d(w)[\frac{d(\xi_1(w), \xi_2(w)) + d(\xi_2(w), \xi_2(w)) + d(\xi_1(w), \xi_2(w))}{1 + d(\xi_1(w), \xi_2(w)) d(\xi_2(w), \xi_2(w)) d(\xi_1(w), \xi_2(w))}] \\
& + \frac{e(w)}{2}[\frac{d(\xi_2(w), \xi_2(w)) + d(\xi_2(w), \xi_3(w)) + d(\xi_1(w), \xi_3(w))}{1 + d(\xi_2(w), \xi_2(w)) d(\xi_2(w), \xi_3(w)) d(\xi_1(w), \xi_3(w))}]
\end{aligned}$$

$$\begin{aligned}
d(\xi_2(w), \xi_3(w)) &\le k\;\; d(\xi_1(w), \xi_2(w)) \\
&\le k^2 d(\xi_0(w), \xi_1(w))
\end{aligned}$$

Similarly, proceding in the same way, by induction we produce a sequence of measurable mapping

$\xi_n : \Omega \to X$ *suchthat for* $\gamma \rangle 0$ *and any* $w \in \Omega$,

$$\xi_{2\gamma+1}(w) \in S(w, \xi_{2\gamma}(w)),\, \xi_{2\gamma+2}(w) \in T(w, \xi_{2\gamma+1}(w))$$

$$and\; d(\xi_n(w), \xi_{n+2}(w)) \le k\;\; d(\xi_{n-1}(w), \xi_n(w)) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \le k^n d(\xi_0(w), \xi_1(w))$$

Furthermoe for $m > n$

$$\begin{aligned}
d(\xi_n(w), \xi_m(w)) &\le\;\; d(\xi_n(w), \xi_{n+1}(w)) + d(\xi_{n+1}(w), \xi_{n+2}(w)) + \ldots\ldots\ldots\ldots + d(\xi_{m-1}(w), \xi_m(w)) \\
&\le [k^n + k^{n-1} + \ldots\ldots\ldots + k^{m-1}] d(\xi_0(w), \xi_1(w)) \\
&\le d(\xi_0(w), \xi_1(w)) k^n [1 + k + k^2 + \ldots\ldots + k^{m-n-1}]
\end{aligned}$$

$$d(\xi_n(w), \xi_m(w)) \le \frac{k^n}{1-k} d(\xi_0(w), \xi_1(w)) \to 0\; as\; m, n \to \infty.$$

It follows that $\{\xi_n(w)\}$ is a Cauchy sequence and there exists a measurable mapping $\xi : \Omega \to X$ such that $\xi_n(w) \to \xi(w)$ for each $w \in \Omega$. It further implies that $\xi_{2\gamma+1}(w) \to \xi(w)$ and $\xi_{2\gamma+2}(w) \to \xi(w)$.

Thus we have for any $w \in \Omega$,

$$d(\xi(w),S_i(w,\xi(w)))\le d(\xi(w),\xi_{2\gamma+2}(w))+d(\xi_{2\gamma+2}(w),S_i(w,\xi(w)))$$
$$\le d(\xi(w),\xi_{2\gamma+2}(w))+H(T_j(w,\xi_{2\gamma+1}(w)),S_i(w,\xi(w))$$
$$d(\xi(w),S_i(w,\xi(w)))\le d(\xi(w),\xi_{2\gamma+2}(w))$$
$$+a(w)d(\xi_{2\gamma+1}(w),\xi(w))+b(w)[d(\xi_{2\gamma+1}(w),T_j(w,\xi_{2\gamma+1}(w)))+d(\xi(w),S_i(w,\xi(w)))]$$
$$+\frac{c(w)}{2}[d(\xi_{2\gamma+1}(w),S_i(w,\xi(w)))+d(\xi(w),T_j(w,\xi_{2\gamma+1}(w)))]$$
$$+d(w)[\frac{d(\xi_{2\gamma+1}(w),\xi(w))+d(\xi(w),T_j(w,\xi_{2\gamma+1}(w)))+d(\xi_{2\gamma+1}(w),T_j(w,\xi_{2\gamma+1}(w)))}{1+d(\xi_{2\gamma+1}(w),\xi(w))d(\xi(w),T_j(w,\xi_{2\gamma+1}(w)))d(\xi_{2\gamma+1}(w),T_j(w,\xi_{2\gamma+1}(w)))}]$$
$$+\frac{e(w)}{2}[\frac{d(\xi(w),T_j(w,\xi_{2\gamma+1}(w)))+d(\xi(w),S_i(w,\xi(w)))+d(\xi_{2\gamma+1}(w),S_i(w,\xi(w)))}{1+d(\xi(w),T_j(w,\xi_{2\gamma+1}(w)))d(\xi(w),S_i(w,\xi(w)))d(\xi_{2\gamma+1}(w),S_i(w,\xi(w)))}]$$

$Letting\,\gamma\to\infty, we\,have$

$$d(\xi(w),S_i(w,\xi(w)))\le 0$$

$Hence\,\xi(w)\in S_i(w,\xi(w))\,for\,w\in\Omega.\,Similarly\,for\,any\,w\in\Omega,$

$$d(\xi(w),T_j(w,\xi(w)))\le d(\xi(w),\xi_{2\gamma+1}(w))+H(S_i(w,\xi_{2\gamma+1}(w)),T_j(w,\xi(w))$$
$$d(\xi(w),T_j(w,\xi(w)))\le 0$$

$Therefore\,\xi(w)\in T_j(w,\xi(w))\,for\,w\in\Omega.$

Corollary 1: let X be a polish space and T:Ω×X →C (X) be a continuous random multivalued operator. If there exists a measurable map a,b:Ω→(0,1) such that for each x,y ∈X and w ∈Ω,

$$H(T_j(w,x),T_j(w,y))\le a(w)d(xy)+b(w)[d(x,T_j(w,x))+d(y,T_j(w,y))]$$
$$+\frac{C(w)}{2}[d(x,T_j(w,y))+d(y,T_j(w,x))]+d(w)[\frac{d(x,y)+d(y,T_j(w,x))+d(x,T_j(w,x))}{1+d(x,y)d(y,T_j(w,x))\,d(x,T_j(w,x))}]$$
$$+\frac{e(w)}{2}[\frac{d(y,T_j(w,x))+d(y,T_j(w,y))+d(x,T_j(w,y))}{1+d(y,T_j(w,x))+d(y,T_j(w,y))+d(x,T_j(w,y))}]\;for\,each\,x,y\in X,w\in\Omega$$

$and\,a,b,c,d,e\in R^+\,with\,2[a(w)+c(w)]+4[b(w)+d(w)]+3e(w)<2;\,j=1,2..n..\infty.$

Corollary 2: Let X be Polish Space. Let T, S: Ω × X → C(X) be two continuous random multivalued operators. If there exists measurable mappings a,b,c,d,e: Ω →(0,1), such that

$$H(S(w,x),T(w,y)) \le a(w)d(x,y)+b(w)[d(x,S(w,x))+d(y,T(w,y))]+\frac{c(w)}{2}[d(x,T(w,y))+d(y,S(w,x))]$$
$$+d(w)[\frac{d(x,y)+d(y,S(w,x))+d(x,S(w,x))}{1+d(x,y)\,d(y,S(w,x))\,d(x,S(w,x))}]$$
$$+\frac{e(w)}{2}[\frac{d(y,S(w,x))+d(y,T(wy))+d(x,T(wy))}{1+d(y,S(w,x))+d(y,T(wy))+d(x,T(wy))}] \; for\,each\,x,y\in X, w\in\Omega$$

$and\, a,b,c,d,e\in R^{+}\; with\; 2[a(w)+c(w)]+4[b(w)+d(w)]+3e(w)<2,$

then there exists a common fixed point of S and T (Here H represents the hausdroff metric on C (X) induced by the metric d)

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