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A Fixed Point Theorems in Cone Metric Spaces with W-Distance

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Abstract

Combining the notion of a metric space with w-distance and the notion of a cone metric space, Lakzian and Arabyani [2] have introduced the notion of a cone metric space with w- distance and proved a fixed point theorem where in the concept of 'infimum' of set is used which may not be meaning in cone metric spaces. In this paper we avoided this difficulty.

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Introduction

Huang Long-Guang and Zhang Xian [1] have introduced Cone metric space . Metric space with $w-$distance was introduced by Osama Kada, Tomonari Suzuki, Wataru Takahashi, and Naoki Shioji [3,4]. These two concepts were combined together and Cone metric space with $w-$distance was introduced by H.Lakzian and F.Arabyani [2], and a fixed point theorem was proved. But in this paper 'infimum' of a set is used which may not be meaningful in the context of a cone metric space.we successfully avoided this difficulty in this paper by obtaining suitable modifications.

Preliminaries

Let E be a real Banach space and P a subset of E. P is called a cone if

1. P is closed, non-empty and $P \neq \{0\}$,
2. $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,
3. $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P. x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is called normal if there is a number $M > 0$ such that for all

$$x, y \in E, 0 \leq x \leq y \; implies \; \|x\| \leq M\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P ([1]). The cone P is called regular if every increasing sequence which is bounded form above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \ldots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty}\|x_n - x\| = 0$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It has been mentioned that every regular cone is normal [1].

Lemma2.1: [1] Every regular cone is normal.
In the following we always suppose that E is a Banach space, P is a cone in E with $int\, P \neq \emptyset$ and $\leq$ is partial ordering with respect to P.

Definition2.2: Let X be a non-empty set. Suppose the mapping $d: X \times X \to E$ satisfies:

$0 \leq d(x, y) \; for \; all \; x, y \in X \; and \; d(x, y) = 0 \; if \; and \; only \; if \; x = y,$
$d(x, y) = d(y, x) \; for \; all \; x, y \in X,$
$d(x, y) \leq d(x, z) + d(z, y) \; for \; all \; x, y, z \in X.$

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example2.3: [1] Let$E = R^2$, $P = \{(x, y) \in E \,/\, x, y \geq 0\}, X = R$ and $d: X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$ where $\alpha \geq 0$ is constant. Then (X, d) is a cone metric space.

Definition2.4: [3] Let X be a cone metric space with metric d. Then a mapping $p: X \times X \to E$ is called w−distance on X if the following satisfy:

$0 \leq p(x, y) \; for \; all \; x, y \in X,$
$p(x, z) \leq p(x, y) + p(y, z) \; for \; all \; x, y, z \in X$
$p(x, .) \to E \; is \; lower \; semi \; continuous \; for \; all \; x \in X$

for any $0 \ll \alpha$, there exists $0 \ll \beta$ such that $p(z,x) \ll \beta$ and $p(z,y) \ll \beta$ imply $d(x,y) \ll \alpha$ where $\alpha, \beta \in E$.

Example 2.5: [2]

1. Let (X, d) be a metric space. Then $p = d$ is a w-distance on X.
2. Let X be a norm linear space with Euclidean norm. Then the mapping $p: X \times X \to [0, \infty)$ defined by $p(x,y) = ||x|| + ||y||$ for all $x, y \in X$ is a w-distance on X.

3. Let X be a norm linear space with Euclidean norm. Then the mapping $p: X \times X \to [0, \infty)$ defined by $p(x, y) = ||y||$ for all $x, y \in X$ is a w-distance on X.

4. Let X be a cone metric space with metric d, p be a w−distance on X and f a function from X into E such that $0 \leq f(x)$ for any $x \in X$. Then a function $q: X \times X \to E$ given by $q(x, y) = f(x) + p(x, y)$ for each $(x, y) \in X \times X$ is also a w−distance.

Definition2.6: [2] Let X be a cone metric space with metric d, let p be a w−distance on X, $x \in X$ and $\{x_n\}$ a sequence in X, then $\{x_n\}$ is called a p−Cauchy sequence whenever for every $\alpha \in E, 0 \ll \alpha$, there is a positive integer N such that, for all $m, n \geq N, p(x_m, x_n) \ll \alpha$.

A sequence $\{x_n\}$ in X is called a p−convergent to a point $x \in X$ whenever for every $\alpha \in E$, $0 \ll \alpha$, there is a positive integer N such that for all $n \geq N, p(x, x_n) \ll \alpha$.

Note that by lower semi-continuous p, for all $n \geq N, p(x, x_n) \ll \alpha$.

We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

(X, d) is a complete cone metric space with w−distance if every Cauchy sequence is p−convergent.

We note that $intP + intP \subseteq intP \text{ } and \text{ } \lambda \, intP \subseteq intP \; (\lambda > 0)$.

Lemma2.7: [1] There is not normal cone with normal constant $M < 1$.

Example2.8: [2] Let $E = l^1$, $P = \{\{x_n\}_{n\geq 1} \in E: x_n \geq 0$, for all n$\}$, (X, p) a metric space and $d : X \times X \to E$ defined by $d(x,y) = \left\{\frac{p(x,y)}{2^n}\right\}_{n\geq 1}$. Then (X, d) is a cone metric space .If we set $p = d$ then (X, d) is cone metric space with w−distance p and the normal constant of P is equal to 1.

Proposition2.9: [1] For each $k > 1$, there is a normal cone with normal constant $M > k$.

Theorem2.10: [2] Let (X, d) be a cone metric space with w−distance p on X and $T: X \to X$. Suppose that there exists $r \in [0, 1)$ such that $p(Tx, T^2x) \leq rp(x, Tx)$, for every $x \in X$ and

$$inf \, \{ \, p(x,y) \, + \, p(x,Tx) \, / \, x \, \in \, X\} \, > \, 0 \tag{2.10.1}$$

for every $y \in X$ with $y \neq Ty$. Then there is a $z \in X$ such that $z = Tz$. Moreover, if P be a normal cone with normal constant M and $v = T(v)$, then $p(v, v) = 0$.

In the cone metric space context 'inf' in (2.10.1) may not be meaningful.

In this paper this difficulty is successfully avoided by suitably modifying the hypothesis (Theorem 3.2). Some supporting examples are also given.

Remark2.11: [2] The sentence "$v = Tv$ then $p(v, v) = 0$" makes the previous applicable theorem, i.e. when we search the fixed point T, we must search the points v where $p(v, v) = 0$; because if $p(v, v) \neq 0$ then $v \neq Tv$.

Theorem2.12: [2] Let (X, d) be a complete cone metric space with w−distance p. Let P be a normal cone on X. Suppose a mapping $T: X \to X$ satisfies the contractive condition $p(Tx, Ty) \leq k\, p(x, y)$, for all $x, y \in X$, where $k \in [0, 1)$ is a constant. Then, T has a unique fixed point in X. For each $x \in X$, the iterative sequence $\{T^n(x)\}_{n\geq 1}$ converges to the fixed point.

Main Results

Lemma3.1: Let X be a cone metric space with metric d and $p : X \times X \to E$ is w-distance on X satisfies $p(Tx, T^2x) \leq rp(x, Tx)$ $\forall$ $r \in [0,1)$ then $\{x_n\}$ is a Cauchy sequence.

Theorem3.2: Let (X, d) be a cone metric space with w−distance p on X and $T: X \to X$. Suppose that there exists $r \in [0, 1)$ such that $p(Tx, T^2x) \leq rp(x, Tx)$, for every $x \in X$ and that $y \neq Ty$. Then there exists $s_y > 0$ such that $0 < s_y \leq \{p(x, y) + p(x, Tx) \,/\, x \in X\} > 0$. Then there is a $z \in X$ such that $z = Tz$. Further if $v = T(v)$, then $p(v, v) = 0$.

Proof: Let $u \in X$ and define $u_n = T^n u$ for any $n \in N$. Then we have, for any $n \in N$, $p(u_n, u_{n+1}) \leq rp(u_{n-1}, u_n) \leq \cdots \leq r^n p(u_0, u_1)$, $0 < r < 1$.

So if $m > n$,

$$p(u_n, u_m) \leq p(u_n, u_{n+1}) + \cdots + p(u_{m-1}, u_m)$$
$$\leq r^n p(u_0, u_1) + \ldots + r^{m-1} p(u_0, u_1)$$
$$\leq \frac{r^n}{1-r} p(u_0, u_1).$$

Now let $\alpha \in E$ with $0 \ll \alpha$ be given, then choose $y \in E$ with $0 \ll y$ such that $\alpha + N_y(0) \subseteq P$ where $\mathrm{N_y}(0) = \{z \in X / ||z|| < y\}$.

Also choose a natural number N_1 such that $\frac{r^n}{1-r} p(u_0, u_1) \in N_{\frac{\varepsilon}{2}}(0)$ then $\frac{r^n}{1-r} p(u_0, u_1) \ll \alpha$

$$\Rightarrow \alpha - \frac{r^n}{1-r} p(u_0, u_1) \in int\, p$$

$\therefore\ p(u_n, u_m) \leq \frac{r^n}{1-r} p(u_0, u_1) \ll \alpha\ \forall\ n > m..$

$\therefore \{u_n\}$ is a Cauchy sequence in X (by Lemma 3.1)

Since X is complete,$\{u_n\}$ converges to some point $z \in X$. Let $n_0 \in N$ be fixed.

Then since $\{u_m\}$ converges to z and $p(u_{n_0}, .)$ is continuous, we have $p(u_{n_0}, z) \leq lim_{m\to\infty} p(u_{n_0}, u_m)$

$\leq \frac{r^{n_0}}{1-r} p(u_0, u_1).$

Assume that $z \neq Tz$. Then by hypothesis we have,

$0 < s_z \leq \{p(x,y) + p(x,Tx) \,/\, x \in X\}$

$0 < s_z \leq \{ p(u_{n_0}, z) + p\big(u_{n_0}, u_{n_0+1}\big)/n_0 \in N\}$

$\leq \{ \left(\frac{r^{n_0}}{1-r}\right) p(u_0, u_1) + r^{n_0} p(u_0, u_1) \}\ \ \forall\, n_0 \in N.$

$\to 0$ as $n_0 \to \infty$

$\Rightarrow\ 0 < s_z \leq 0$

This is a contradiction.

$\therefore\ z = Tz.$

$If\ v = Tv$ we have $p(v,v) = p(Tv, T^2v)\ r\ p(v, Tv) = r\ p(v,v).$

$\therefore\ p(v,v) \leq r\ p(v,v)$

$\Rightarrow (1-r)\ p(v,v) \leq 0$

$\Rightarrow\ p(v,v) = 0.$

Example3.3: Let (X,d) be a complete cone metric space with w-distance P on X and the mapping $T : X \to X$. Suppose that there exists $r \in [0, 1)$ such that $p(Tx, T^2x) \leq rp(x,Tx)$ for every $x \in X$ and that $y \neq Ty$. Then there exists $s_y > 0$ such that $0 < s_y = y - y^2 \leq d(x,y) + d(x,x^2)$ for every $x \in [0,\frac{1}{2}]$ and $y \in (0,1)$ and $d(x,y) = |x-y|. \ni T(x) = x^2$ on $X = R$.

Then there is a $z \in X$ such that $z = Tz$. Further if $v = T(v)$, then $p(v,v) = 0$.

Example3.3: Let (X, d) be a complete cone metric space with w-distance P on X and the mapping $T : X \to X$. Suppose that there exists $r \in [0,1)$ such that $p(Tx, T^2x) \leq rp(x,Tx)$, for every $x \in X$ and that $y \neq Ty$.

Then there exists $s_y > 0$ such that $0 < s_y \leq \{p(x,y) + p(x,Tx)\ /\ x \in X\}$ where $p(x,y) = (|x-y|, \frac{1}{2}|x-y|)$ and $X\colon R \to R^2$. Then there is a $z \in X$ such that $z = Tz$. Moreover, if p be a normal cone with normal constant M and $v = T(v)$, then $p(v,v) = 0$.

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