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Common Fixed Point Theorems in Menger Spaces

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Abstract

In this paper Theorem 3.1 of Kubiaczyk and Sushil Sharma [5] is shown to hold even under weaker hypothesis (Theorem 2.2) and we obtain a fixed point theorem (Theorem 2.3) involving occasionally weakly compatible maps and also prove a coincidence point theorem (Theorem 2.4) for a pair of self maps under certain conditions. Examples are provided to show that the hypothesis in Theorems 2.3 and 2.4 can not be further weakend.

Keywords: common fixed point, compatible maps, weakly compatible maps, occasionally weakly compatible maps.

Mathematical subject classification (2000): 47H10, 54H25

Introduction

In this paper, we show that Theorem 3.1 of Kubiaczyk and Sushil Sharma [5] holds even if some hypothesis is dropped.

Definition 1.1: [7] A function $F: R \to [0,1]$ is called a distribution function if

1. F is non-decreasing,
2. F is left continuous,
3. $\inf_{x\in \mathrm{R}} F(x) = 0 \ and \ \sup_{x\in \mathrm{R}} F(x) = 1$

Definition 1.2: [7] Let X be a non-empty set and $F: X \times X \rightarrow \mathfrak{D}$ (The set of distribution functions). For, $p, q \in X$, we denote the image of the pair (p, q) by $F_{p,q}$ which is a distribution function so that $F_{p,q}(x) \in [0,1]$, for every real x.

Suppose F satisfies:

1. $F_{p,q}(x) = H(t)\ if\ and\ only\ if\ p = q$
2. $F_{p,q}\ (0) = 0$
3. $F_{p,q}\ (x) = F_{q,p}(x)$
4. If $F_{p,q}\ (x) = 1$ and $F_{q,r}\ (y) = 1$ then $F_{p,r}(x + y) = 1$ where $p, q, r \in X$.

Then (X, F) is called a probabilistic metric space.

Definition 1.3: [7] A triangular norm $t : [0,1]\, x\, [0,1] \rightarrow [0,1]$ is a function satisfying the following conditions

1. $t(\alpha, 1) = \alpha \quad \forall\, \alpha \in [0,1]$
2. $t(\alpha, \beta) = t(\beta, \alpha) \quad \forall\, \alpha, \beta\, \epsilon\, [0,1]$
3. $t(\gamma, \delta) \geq t(\, \alpha, \beta) \ \forall\, \alpha, \beta, \gamma, \delta\, \epsilon\, [0,1]\ with\, \gamma \geq \alpha\ and\ \delta \geq \beta$
4. $t(t(\alpha, \beta), \gamma) = t(\alpha, t(\beta, \gamma))\ \forall\, \alpha, \beta, \gamma \in [0,1]$

Definition 1.4: [7] A Menger probabilistic metric space (X, F, t) is an ordered triad, where t is a t-norm and (X, F) is probabilistic metric space satisfies the following: $F_{x,z}(r + s) \geq\ t\left(F_{x,y}(r), F_{y,z}(s)\right)\ \ \forall\, x, y, z \in X\ \ and\, r, s \geq 0.$

Definition 1.5: [4] Two self mappings f and g of a probabilistic semi-metric space (X, F) are said to be Occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

Definition 1.6: [6] Let (X, F, t) is a Menger space with the continuous T-norm t. A sequence $\{p_n\}$ in X is said to be convergent to a point $p \in X$ if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer $N = N(\varepsilon, \lambda)$ such that $p_n \in U_p(\varepsilon, \lambda)$ for all $n \geq N$, or equivalently, $F_{p,p_n}(\varepsilon) > 1 - \lambda$, for all $n \geq N$.

We write $p_n \rightarrow p$ as $n \rightarrow \infty$ or $\lim_{n\rightarrow\infty} p_n = p$

Definition 1.7: [2] Let (X, F, t) be as Menger space such that the T-norm. t is continuous and S, T be mappings from X into itself. Then S and T are said to be compatible if $\lim_{n\rightarrow\infty} F_{STx_n,TSx_n}(x) = 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $lim_{n\rightarrow\infty} Sx_n = lim_{n\rightarrow\infty} T\ x_n = z$ for some $z \in X$.

Definition 1.8: [3] Two self mappings S and T are said to be weakly compatible if they commute at their coincidence points if $Tu = Su$ for some $u \in X$ then $STu = TSu$.

Definition 1.9: [1] Let S and T be two self mappings of a Menger space (X, F, t). We say that S and T satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X

$lim_{n\to\infty} S\ x_n = lim_{n\to\infty} T\ x_n = z$ for some $z \in X$.

Main Results

The following Theorem is proved in [5].

Theorem 2.1 ([5], Theorem 3.1): Let (X, F, t) be a Menger space with $t(x,y) = \{x,y\}\ \forall\, x,y \in [0,1]$ and S and T be weakly compatible mappings of X into itself such that S and T satisfy the property (E.A), there exists a number $k \in (0,1)$ such that

$$F_{Tu,Tv}(kx)\ \geq min\{F_{Su,Sv}(x), F_{Su,Tu}(x), F_{Sv,Tv}(x), F_{Sv,Tu}(x), F_{Su,Tv}(x)\}$$

for all $u, v \in X$,

$$T(X) \subset S(X).$$

If S(X) or T(X) be a closed subset of X, then S and T have a unique common fixed point.

It can be shown that Theorem 2.1 holds even when condition (iii) is dropped.

This is the essence of the following theorem.

Theorem 2.2: Let (X, F, t) be a Menger space with $t(x,y) = \{x,y\}\ \forall\, x,y \in [0,1]$, S and T be weakly compatible mappings of X into itself such that

1. S and T satisfy the property (E.A)
2. there exists a number $k \in (0,1)$ such that

$$F_{Tu,Tv}(kx) \geq min\{F_{Su,Sv}(x), F_{Su,Tu}(x), F_{Sv,Tv}(x), F_{Sv,Tu}(x), F_{Su,Tv}(x)\}$$

for all $u, v \in X$ and for $x > 0$.

If S(X) be a closed subset of X, then S and T have a unique common fixed point.

Proof: Since S and T satisfy the property (E.A), there exists a sequence $\{x_n\}$ in X satisfying $lim_{n\to\infty} S\ x_n = lim_{n\to\infty} T\ x_n = z$ for some $z \in X$

Suppose that S(X) is closed.

Then $lim_{n\to\infty} Sx_n\ = Sa$ for some $a \in X$, also $\lim_{n\to\infty} Tx_n\ = Sa$.

By condition (ii), for $x > 0$, we have

$F_{Tx_n,Ta}(kx) \geq \min\ \{ F_{Sx_n,Sa}(x), F_{Sx_n,Tx_n}(x), F_{Sa,Ta}(x), F_{Sa,Tx_n}(x), F_{Sx_n,Ta}(x)\}$

Letting $\to \infty$, we get

$$F_{Sa,Ta}(kx)\ \geq\ F_{Sa,Ta}(x) \geq\ F_{Sa.Ta}(kx)$$

$$\therefore F_{Sa,Ta}(t)\ =\ F_{Sa,Ta}(x)\ \ \forall\ \ t\ \in [kx, x] \text{ for } x > 0.$$

Similarly, for any $t \in [k^2x, kx]$,

$$F_{Sa,Ta}(t) = F_{Sa,Ta}(kx) =\ F_{Sa,Ta}(x)$$

In general, we can prove that

$$F_{sa,Ta}(t) =\ F_{Sa,Ta}(x)\ \forall\, t\ \in [\ k^n x,\ x]$$

Or equivalently, $F_{sa,Ta}(t) = F_{Sa,Ta}(x) \ \forall\, t \in \left[x, \frac{x}{k^n}\right]$ for $n = 1,2,3$ ....
Hence

$$F_{Sa,Ta}(x) = 1 \ \forall\, x > 0$$
$$\therefore Ta = Sa.$$

Since S and T are weakly compatible $STa = TSa$.

$$\therefore TTa = TSa = STa = SSa$$

We prove that Ta is a common fixed point of T and S.
Consider, for $x > 0$,
$F_{Ta,TTa}(kx) \geq \min\{F_{Sa,STa}(x), F_{Sa,Ta}(x), F_{STa,TTa}(x), F_{STa,Ta}(t), F_{Sa,TTa}(x)\}$
$= F_{Ta,TTa}(x)$
Hence $F_{Ta,TTa}(kx) \geq F_{Ta,TTa}(x)$.
So that $F_{Ta,TTa}(t) = F_{Ta,TTa}(x) \ \forall\, t \in [kx, x]$.
Consequently, $F_{Ta,TTa}(t) = 1 \ \forall\ t > 0$.
$\therefore Ta = TTa$
Hence $TTa = Ta$ and $STa = TTa = Ta$
i.e. Ta is common fixed point for S and T.
Uniqueness of the common fixed point follows easily.

Theorem 2.3: Let (X, F, t) be a Menger space with $t(x,y) = \min\{x,y\}$ *for all* $x, y \in [0,1]$, S and T be occasionally weakly compatible maps of X satisfying there exists a number $k \in (0,1)$ such that

$$F_{Tu,Tv}(kx) \geq min\{F_{Su,Sv}(x), F_{Su,Tu}(x), F_{Sv,Tv}(x), F_{Sv,Tu}(x), F_{Su,Tv}(x)\}$$
$$for\ all\ u, v \in X.$$

Then S and T have a unique common fixed point.

Proof: Since S, T are occasionally weakly compatible maps there exists $a \in X$ such that $Sa = Ta$ and $STa = TSa$.

Suppose $Sa = Ta = w$ and $Sb = Tb = w'$ be two points of coincidence, then by (i), we have

$$F_{Tu,Tv}(kx) \geq min\{F_{Su,Sv}(x), F_{Su,Tu}(x), F_{Sv,Tv}(x), F_{Sv,Tu}(x), F_{Su,Tv}(x)\}$$
$$\Rightarrow F_{w,w'}(kx) = F_{Ta,Tb}(kx) \geq F_{Ta,Tb}(x)$$
$$\Rightarrow F_{w,w'}(kx) \geq 1$$
$$\Rightarrow w = w'$$

Since S, T are occasionally weakly compatible and have unique point of coincidence, then have w is a unique common fixed point of S and T.

Theorem 2.4: Let (X, F, t) be a Menger space with $t(x,y) = \min\{x,y\}$ for all $x, y \in [0,1]$, S and T be self mappings of X into itself such that

1. S and T satisfy the property (E.A)

2. there exist a number $k \in (0,1)$ such that
$F_{Tu,Tv}(kx) \geq min\{F_{Su,Sv}(x), F_{Su,Tu}(x), F_{Sv,Tv}(x), F_{Sv,Tu}(x), F_{Su,Tv}(x)\}$
$for\ all\ \ u, v \in X$

If S(X) be a closed subset of X, then S and T have a coincidence point.

Proof: Since S and T satisfy the property (E.A) there exist a sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} Sx_n = lim_{n\to\infty} T\, x_n = z$ for some $z \in X$.
Suppose that S(X) is closed.
Then $\lim_{n\to\infty} Sx_n = Sa$ for some $a \in X$, also $\lim_{n\to\infty} Tx_n = Sa$.
By taking $u = x_n\ and\ v = a$ in condition (5.13.2), we get
$F_{Tx_n,Ta}(kx) \geq \min\{F_{Sx_n,Sa}(x), F_{Sx_n,Tx_n}(x), F_{Sa,Ta}(x), F_{Sa,Tx_n}(x), F_{Sx_n,Ta}(x)\}$

On letting $n \to \infty$, we get
$F_{Sa,Ta}(kx) \geq F_{Sa,Ta}(x) \geq F_{Sa.Ta}(kx)$
$\therefore F_{Sa,Ta}(t) = F_{Sa,Ta}(x)\ \forall\ t \in [kx, x] for\ x > 0$

For any $t \in [k^2x, kx]$,
$F_{Sa,Ta}(t) = F_{Sa,Ta}(x)$

In general we can prove that
$F_{sa,Ta}(t) = F_{Sa,Ta}(x)\ \forall\ x \in [\,x, \frac{x}{k}]$
$F_{sa,Ta}(t) = F_{Sa,Ta}(x)\ \forall\ x \in [\,\frac{x}{k}, \frac{x}{k^2}]$

Continuing this way we can prove that
$F_{Sa,Ta}(x) = 1\ \ \forall\, x > 0$
$\therefore Ta = Sa.$
Hence S and T have a coincidence point.
Theorem 2.2 is not valid if S and T are not weakly compatible in view of the following example:
Example 2.5: Let $X = \{-1, 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\dots, 0, -\frac{1}{2}, \dots\dots, -\frac{(2^n-1)}{2^n}, \dots\}$

Define S, T on X as: $S(1) = 0,\ S(-1) = -1,\ S\left(\frac{1}{n}\right) = \frac{1}{n+1}\ ,\ n \geq 2;$
$S(0) = \frac{-1}{2},\ \dots,\ S(-\frac{(2^n-1)}{2^n}) = \frac{-(2^{n+1}-1)}{2^{n+1}},\ \dots\dots$

And $T(x) = 0\ \ \forall\, x$
Then $S(X) = X - \{\,1, \frac{1}{2}\},\ \ T(X) \subseteq S(X),$
$T(X)$ is closed and SX is closed.
Since $T\left(\frac{1}{n}\right) \to 0, S\left(\frac{1}{n}\right) = \frac{1}{n+1} \to 0$ and therefore property (E.A) holds.
$F_{0,1}(t) = 1, t = kx$ and condition (ii) holds.

But $S1 = T1$ and $ST1 = S0 = -\frac{1}{2},\ TS1 = T0 = 0$

Therefore S and T are not weakly compatible.

The following example shows that Theorem 2.3 may not be valid if occasional weak compatibility is dropped.

Example 2.6: Let $X = \{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots, 0, -\frac{1}{2}, \dots, -\frac{(2^2-1)}{2^2}, \dots, -\frac{(2^n-1)}{2^n}, \dots\}$

Define S by $S\left(\frac{1}{n}\right) = \frac{1}{n+1}\ for\ n \geq 2,\ S0 = -\frac{1}{2}, \dots.$

$S\left(-\frac{(2^n-1)}{2^n}\right) = -\frac{(2^{n+1}-1)}{2^{n+1}},\ \dots..$ and $S1 = 0$

Then $S(X) = X - \left\{1. \frac{1}{2}\right\}$,

Define T by $Tx = 0\ \forall x.$

Then property (E.A) holds,

since $T\left(\frac{1}{n}\right) \to 0, S\left(\frac{1}{n}\right) = \frac{1}{n+1} \to 0$

since $F_{0,0}(kx) = 1$ and condition (i) holds.

$S1 = 0 = T1, TS1 = T0 = 0, ST1 = S0 = \frac{1}{2}$

Hence S and T are not occasionally weakly compatible

Clearly S and T do not have a common fixed point.

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