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Results on Maximization Theorem of Analytic Functions Related to Complex Order

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Abstract

Purpose of this paper is to introduce a class G (λ, μ, A, B, b) of a analytic functions f(z) of complex order b via convolution technique. In our result a maximization theorem is proved which is generalization of Choudhary [1]. On the other hand in the next result a convolution condition is given in which by taking μ =1, A=1, B=-1, we obtain another result of Choudhary [1]. To meet our requirement a lemma is also proved.

Mathematics subject classification: 30C45, 30C50

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Introduction

The analytic functions have played a very important role in the development of certain subclasses of analytic functions of a complex order. Many authors obtained some beautiful results regarding subclasses of analytic functions. To see this end we refer Chaudhary [1], Silvia [2], Keogh and Merkes [3] Silverman [4], Kim and Show [5], Shaqsi and Darus [6], Nasr and Aouf [7], Ahuja [8], Chichra [9], Goel and Mehrok [10], Shukla and Dashrath [11] and Ruscheweyh [12].

Let f(z) be an analytic function in class A such that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ where f (z) ∈ A is analytic and univalent in the unit disk U = {z: |z |< 1}.

In this paper, we introduce a class G (λ, μ, A, B, b) of analytic functions f(z) of complex order b, by using convolution technique, as follows. A function f of A belongs to the class G (λ, μ, A, B, b) if and only if there exists a function w belonging to the class H such that

$$1+\frac{1}{b}\left\{\frac{z(D^{\lambda} f(z)}{D^{\lambda} f(z)}-1\right\}=(1-\mu)+\mu\left\{\frac{1+Aw(z)}{1+Bw(z)}\right\},\ z \in \mathrm{U} \qquad (1.1)$$

where $-1 \le B < A \le 1$, $0 < \mu \le 1$, $\lambda > -1$,

Preliminaries

Def. (2.1): Let f (z) defined by $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ and b is any non-zero complex number then D^{λ} f (z) defined by

$$D^{\lambda} f(z)=\frac{z}{(1-z)^{\lambda+1}} * f(z)=\frac{z(z^{\lambda-1} f(z))^{(\lambda)}}{\lambda!}$$

where * denotes the Hadamard product of two analytic functions.

Def. (2.2): If f (z) and g (z) are any two functions in class A such that

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z)=z+\sum_{n=2}^{\infty} b_n z^n$$

then the convolution or Hadamard product of f (z) and g (z) is denoted by f * g, and is defined by the power series

$$(f * g)(z)=z+\sum_{n=2}^{\infty} a_n b_n z^n .$$

Now using the identity $z(D^{\lambda} f(z))'=(\lambda+1) D^{\lambda+1} f(z)-\lambda D^{\lambda} f(z)$ in (1.1), we have

$$1+\frac{(\lambda+1)}{b}\left\{\frac{D^{\lambda+1} f(z)}{D^{\lambda} f(z)}-1\right\}=(1-\mu)+\mu\left\{\frac{1+Aw(z)}{1+Bw(z)}\right\},\ z \in \mathrm{U} \qquad (2.3)$$

Clearly this can be seen that (1.1) and (2.3) are equivalent to

$$\left|\frac{\left\{\frac{z(D^{\lambda} f(z)'}{D^{\lambda} f(z)}-1\right\}}{\mu(A-B)b-B\left\{\frac{z(D^{\lambda+1} f(z)'}{D^{\lambda} f(z)}-1)\right\}}\right|<1,\ z \in \mathrm{U} \qquad (2.4)$$

and

$$\left|\frac{(\lambda+1)\left\{\frac{(D^{\lambda}f(z)}{D^{\lambda}f(z)}-1\right\}}{\mu(A-B)b-(\lambda+1)B\left\{\frac{(D^{\lambda+1}f(z)}{D^{\lambda}f(z)}-1)\right\}}\right|<1,\ z\in U \qquad (2.5)$$

respectively.

Main Results

Before giving our main result we prove the following Lemma and also quote a Lemma due to Keogh and Merkes [3].

Lemma (3.1): For a fixed integer k, k ≥ 3, let

$$M_j=\frac{|\mu(A-B)-(j-2)B|^2}{(\lambda+j-1)^2},\ \ (j=2,3,...\ k)$$

and

$$c(\lambda,p)=\frac{(\lambda+1)_{p-1}}{p-1!}$$

$$c(\lambda,p)=\frac{(\lambda+1)(\lambda+2)....(\lambda+p-1)}{p-1!},\ (p=2,3\ ,...)$$

then

$$\frac{1}{\{(k-1)c(\lambda,k)\}^2}\Big[\mu^2(A-B)^2|b|^2+\sum_{p=2}^{k-1}\{|\mu(A-B)b -(p-1)B|^2-(p-1)^2\}\{c(\lambda,p)\}^2\prod_{j=2}^{p}M_j\Big]=\prod_{j=2}^{k}M_j \qquad (3.1.1)$$

Proof: We shall prove (3.1.1) by mathematical induction on k. It can easily be seen that (3.1.1) holds for k = 3. Now assume that (3.1.1) is valid for k = 4, 5,..., t-1, then for k = t the left side of (3.1.1) gives

$$\frac{1}{\{(t-1)c(\lambda,t)\}^2}\Big[\mu^2(A-B)^2|b|^2+\sum_{p=2}^{t-1}\{|\mu(A-B)b -(p-1)B|^2\ \_(p-1)^2\}\{c(\lambda,p)\}^2\prod_{j=2}^{p}M_j\Big]$$

$$= \frac{1}{\{(t-1)c(\lambda,t)\}^2}\Big[\mu^2(A-B)^2|b|^2 + \sum_{p=2}^{t-2}\{|\mu(A-B)b$$

$$-(p-1)B|^2-(p-1)^2\}\{c(\lambda,p)\}^2\prod_{j=2}^{p}M_j + \{|\mu(A-B)b$$

$$-(t-2)B|^2-(t-2)^2\}\{c(\lambda,t-1)\}^2\prod_{j=2}^{t-1}M_j\Big]$$

$$= \frac{1}{\{(t-1)c(\lambda,t)\}^2}\Big[\{(t-2)c(\lambda,t-1)\}^2\prod_{j=2}^{t-1}M_j$$

$$+\{(\lambda,t-1)^2M_t-(t-2)^2\}\{c(\lambda,t-1)\}^2\prod_{j=2}^{t-1}M_j\Big]$$

$$= \prod_{j=2}^{t}M_j$$

This concludes the proof of the above lemma.

Lemma (3.2) [3]: Let w (z) = $\sum_{k=1}^{\infty} c_k z^k$ be analytic with $|w(z)|<1$ in U. If d is any complex number, then $|c_2 - dc_1^2| \le \max\{1,|d|\}$. Equality may be attained with the function w (z) = z^2 and w (z) = z.

Now we prove our main result. In this result we determine the maximization of $|a_3 - \delta a_2^2|$ for complex value of δ over the class G (λ, μ, A, B, b).

Maximization theorem

Theorem (3.3): If f (z) defined by (1.1) and belongs to the class G (λ, μ, A, B, b) and δ is any complex number, then

$$|a_3 - \delta a_2^2| \le \frac{\mu(A-B)|b|}{2c(\lambda,3)}\max\{1,|d|\}, \tag{3.3.1}$$

where

$$d = \frac{2\delta\mu(A-B)bc(\lambda,3)-\{\mu(A-B)b-B\}\{c(\lambda,2)\}^2}{\{c(\lambda,2)\}^2}$$

The inequality (3.3.1) is sharp for each δ.

Proof: Since the function f (z) belongs to the class G (λ, μ, A, B, b), then we have

$$1+\frac{1}{b}\left\{\frac{z(D^{\lambda}f(z))'}{D^{\lambda}f(z)}-1\right\}=(1-\mu)+\mu\left\{\frac{1+Aw(z)}{1+Bw(z)}\right\} \quad z\in U \qquad (3.3.2)$$

where

$$w(z)=\sum_{k=1}^{\infty}c_k z^k$$

, from (3.3.2) we have

$$w(z)=\frac{z(D^{\lambda}f(z))'-D^{\lambda}f(z)}{\{\mu(A-B)b+B\}D^{\lambda}f(z)-Bz(D^{\lambda}f(z))'}$$

or

$$w(z)=\frac{\sum_{n=2}^{\infty}(n-1)c(\lambda,n)a_n z^n}{\mu(A-B)bz+\sum_{n=2}^{\infty}\{\mu(A-B)b-(n-1)B\}c(\lambda,n)a_n z^n}$$

$$w(z)=\frac{1}{\mu(A-B)b}\left[c(\lambda,2)a_2 z+2c(\lambda,3)a_3 z^2-\frac{\{\mu(A-B)b-B\}}{\mu(A-B)b}\{c(\lambda,2)\}^2 a_2^2 z^2+....\right]$$

equating the coefficients of z and z^2 on both sides, we get

$$a_2=\frac{\mu(A-B)bc_1}{c(\lambda,2)}$$

and

$$a_3=\frac{\mu(A-B)bc_2+\mu(A-B)b\{\mu(A-B)b-B\}c_1^2}{2c(\lambda,3)}$$

Thus, we have

$$a_3-\delta a_2^2=\frac{\mu(A-B)b}{2c(\lambda,3)}\{c_2-dc_1^2\},$$

where

$$d=\frac{2\delta\mu(A-B)bc(\lambda,3)-\{\mu(A-B)b-B\}\{c(\lambda,2)\}^2}{\{c(\lambda,2)\}^2}$$

hence

$$\left|a_3-\delta a_2^2\right|=\frac{\mu(A-B)|b|}{2c(\lambda,3)}\left|c_2-dc_1^2\right|$$

Therefore, by using Lemma (3.2) in the above equation, we have

$$\left|a_3-\delta a_2^2\right|\le\frac{\mu(A-B)|b|}{2c(\lambda,3)}\max\{1,|d|\}$$

Since the inequality (3.2.2) is sharp, so that the inequality (3.3.1) must also be sharp.

Remark: If we take μ=1, A=1 and B = -1, Theorem (3.3) coincides with the corresponding result of Chaudhary [1].

In our next result we find the necessary and sufficient condition, in terms of Convolution for the function f(z) belonging to the class G (λ, μ, A, B, b).

Convolution Condition

Theorem (3.4): A function f(z) belongs to the class G (λ, μ, A, B, b) if and only if

$$f(z)*\left[\frac{-\mu(A-B)bzX+\left[(\lambda+1)+X\left\{\mu(A-B)b+(\lambda+1)B\right\}\right]z^2}{(1-z)^{\lambda+2}}\right]\neq 0 \qquad (3.4.1)$$

in $0<|z|<1$, where $|X|=1$ and $X\neq 1$.

Proof: Let the function f (z) belongs to the class G (λ, μ, A, B, b), then

$$1+\frac{1}{b}\left\{\frac{z(D^{\lambda}f(z))'}{D^{\lambda}f(z)}-1\right\}\neq(1-\mu)+\mu\left\{\frac{1+AX}{1+BX}\right\} \qquad (3.4.2)$$

$|X|=1$ and $X\neq 1$ in $0<|z|<1$, equivalently

$$(1+BX)\left\{z(D^{\lambda}f(z))'-D^{\lambda}f(z)\right\}-b\mu(A-B)XD^{\lambda}f(z)\neq 0,\ \text{in } 0<|z|<1. \qquad (3.4.3)$$

We know that

$$z(D^{\lambda}f(z))'=(\lambda+1)D^{\lambda+1}f(z)-\lambda D^{\lambda}f(z) \qquad (3.4.4)$$

Using (3.4.4) in (3.4.3), we obtain

$$(1+BX)\left\{(\lambda+1)D^{\lambda+1}f(z)-\lambda D^{\lambda}f(z)\right\}$$
$$-\left\{1+BX+b\mu(A-B)X\right\}D^{\lambda}f(z)\neq 0 \text{ in } 0<|z|<1 \qquad (3.4.5)$$

Since

$$D^{\lambda}f(z)=\frac{z}{(1-z)^{\lambda+1}}*f(z)$$
, (3.4.5) reduces to

$$f(z)*\left[\frac{-\mu(A-B)bzX+\left[(\lambda+1)+X\left\{\mu(A-B)+(\lambda+1)B\right\}\right]z^2}{(1-z)^{\lambda+2}}\right]\neq 0$$
,

which is the required convolution condition. The converse part follows easily since all the steps can be retraced back.

Remark: If we take μ=1, λ=1 and B = -1, Theorem (3.4) coincides with the corresponding results of Chaudhary [1].

References

[1] A.M. Chaudhary: On a class of Univalent functions defined by Ruscheweyh derivatives. Soochow J. Math 15 (1989), 143-157.
[2] E.M. Silvia: Subclasses of spiral-like functions Tamkang J. Math. 14 (1983), 161.
[3] F. R. Keogh and E. P. Merkes, Proc. Amer. Math. Soc. 20 (1969), 8-14.
[4] H. Silverman: Convex and Starlike criteria, I.J.M.M.S. 22 (1999), 75-79.
[5] J. A. Kim and K. H. Show: Mapping properties for convolution IJMMS 17(2003), 1083-1091.
[6] K. A. Shaqsi and M. Darus: On Coefficient problems of Certain Analytic functions involving Hadamard Product I.M.S. 1no.34 (2006), 1669-76.
[7] M.A. Nasr and M.K. Aouf: On convex function of complex order, Mansoura Sci. Bull. (1982), 565-585.
[8] O.P. Ahuja: The Bieberbach conjecture and its impact on the development in geometric function theory Math. Chronical 15 (1986) 1-28.
[9] P.N. Chichra: New subclass of the class of close to convex functions, Proc. Amer. Math. Soc. 62 (1977), 37-43.
[10] R. M. Goel and B. S. Mehrok: J. Austral Math. Soc. (Ser.A) 35(1983), 1-17.
[11] S. L. Shukla and Dashrath: On a class of Univalent functions, Soochow J. Math.10 (1984), 117-126.
[12] S.Ruscheweyh: New criteria for Univalent functions, Proc. Of the Amer. Math. Soc., 49 (1975), 109-11.