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A Note on Goldie Near-Rings

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Abstract

If M is a $K-$module with d.c.c. on $K-$subgroups and satisfying the property (P), then it is shown that M has a submodule which is uniform. Further, if M satisfies the Goldie condition, then it is shown that there exists minimal elements $x_1, x_2, \cdots x_n$ in M such that $< x_1 > \oplus < x_2 > \oplus \cdots \oplus < x_n >$ is direct and M is an essential extension of $< x_1 > \oplus < x_2 > \oplus \cdots \oplus < x_n >$.

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Introduction

All near-rings are assumed to be zero-symmetric right near-rings with identity. Throughout this paper near-ring under consideration is denoted by K.

K.C. Chodhury [1, 2, 3] and BH. Satyanarayana [8, 9] obtained some results on Goldie near-rings. In this paper some results on modules in Goldie near-rings are obtained.

The definitions of $K-$module, $K-$subgroups and submodules are as given in Pilz[7]. For the sake of continuity the definitions are given below.

Definition 1.1[10]: Let $(M,+)$ be a group and K be a near-ring such that there exists a mapping $u: K \times M \to M$ satisfying the conditions;

$$(k+k')m = km + k'm$$
$$(kk')m = k(k'm).$$

$1.m = m$ for all $k, k' \in K, m \in M$ and 1 is the identity of K.

Then $(M,+,u)$ is called a $K-$module.

Definition 1.2: A subset N of a $K-$module M is said to be a $K-$subgroup of M if $(N,+)$ is a subgroup with $KN \subseteq N$.

Definition 1.3: A normal subgroup N of M is called a submodule of M if $k(m+n)-km \in N$ for all $m \in M, n \in N$ and $k \in K$.

Definition 1.4: Let M be a module over a near-ring K. M is said to be an essential extension of a non-zero $K-$subgroup N if for every non-zero $K-$subgroup N', $N \cap N' \neq 0$.

If N is essential in M, then we denote it by $N \leq_e M$.

Definition 1.5: A $K-$module M is said to be uniform if it is an essential extension of each of its non-zero $K-$subgroups.

Notation 1.6: If N is a subset of a $K-$module M, then $<N>$ stands for the submodule of M generated by N. And the submodule generated by an element $x \in M$ is denoted by (x) or $<x>$.

We assume that $K-$module M satisfies the property (P): "$<N_1 \cap N_2> = <N_1> \cap <N_2>$ for any two $K-$subgroups N_1 and N_2 of M."

Any near-ring K in which every $K-$subgroup is a submodule of K satisfies this property.

Definition 1.7: A $K-$module is said to satisfy Goldie condition if it cannot contain an infinite direct sum of submodules.

Main results

Theorem 2.1: If a $K-$module M satisfies Goldie condition, then every submodule M' of M contains a sub module N of M. That is, $M' \supset N$ and N is a submodule of M such that N is uniform.

Proof: Suppose M is not uniform. Then there exists non-zero $K-$subgroups N_1 and N_2 of M such that $N_1 \cap N_2 = 0$.

Therefore $<N_1> \cap <N_2> = <0>$.

Let $M_1 = <N_1>$ and $M_1' = <N_2>$.

Then $M_1 \oplus M_1'$ is direct.

If M_1 is not uniform, then there exists as above non-zero $K-$subgroups N_3 and N_4 of M_1 such that $N_3 \cap N_4 = 0$.

Therefore $< N_3 > \cap < N_4 > = < 0 >$.

Put $M_2 = < N_3 >$ and $M_2^{'} = < N_4 >$. Then the sum $M_2^{'} \oplus M_2 \oplus M_1^{'}$ is direct.

Again if M_2 is not uniform, as above there exists submodules M_3 and $M_3^{'}$ of M such that $M_3 \cap M_3^{'} = < 0 >$ and $M_2 \supset M_3$, $M_2 \supset M_3^{'}$.

Hence, the sum $M_1^{'} \oplus M_2^{'} \oplus M_3^{'} \oplus M_3$ is direct.

Repeating the argument, we get a sequence $\{M_n\}$ of submodules of M which are not uniform such that $M_1 \subset M_1 + M_2 \subset M_1 + M_2 + M_3 \subset \cdots$.

But this is a contradiction to Goldie condition.

Because of Goldie condition, after a finite number of steps one gets a submodule which is uniform.

Applying this construction to any submodule of M, we have that for any submodule $M^{'}$ of M, there exists a uniform submodule N of M such that $M^{'} \supset N$. □

Theorem 2.2: If a $K-$module M satisfies Goldie condition, then there exists uniform sub modules $U_1, U_2, \ldots, U_n$ of M such that $U_1 \oplus U_2 \oplus \ldots \oplus U_n$ is direct and M is essential extension of $U_1 \oplus U_2 \oplus \ldots \oplus U_n$.

Proof: By above theorem, M contains a sub module U_1 which is uniform.

If M is not essential extension of N, then there exists a $K-$subgroup N of M such that $U_1 \cap N = < 0 >$.

Therefore $U_1 \cap < N > = 0$, where $< N >$ is the submodule of M generated by N.

Either $< N >$ is uniform or contains a submodule U_2 of M which is uniform.

That is $< N > \supseteq U_2$, U_2 is uniform.

Therefore $U_1 \oplus U_2$ is direct.

Then by Goldie condition, there exists uniform submodules $U_1, U_2, \ldots, U_n$ of M such that $U_1 \oplus U_2 \oplus \ldots \oplus U_n \leq_e M$. □

Theorem 2.3: Let M be a $K-$module with descending chain condition on $K-$subgroups and satisfying the property (P). Then M has a submodule which is uniform.

Proof: If M is not uniform, then there exists $K-$subgroups N_1 and N_2 such that

$$< N_1 > \cap < N_2 > = < 0 >.$$

Put $M_1 = < N_1 >$, then $M \underset{\neq}{\supset} M_1$.

If M_1 is not uniform, again there exists $K-$subgroups $N_1^{'}, N_2^{'}$ such that $M_1 \supset N_1^{'}, M_1 \supset N_2^{'}$,

$N_1^{'} \cap N_2^{'} = 0$.

Therefore $< N_1^{'} > \cap < N_2^{'} > = < 0 >$.

Put $M_2 = < N_1^{'} >$. Then $M \underset{\neq}{\supset} M_1 \underset{\neq}{\supset} M_2$.

By descending chain condition, after a finite number of steps we get a sub module U of M such that $M \supset U$ and U is uniform. □

Corollary 2.4: If M has descending chain condition on $K-$subgroups, then for any submodule N of M, there exists a uniform submodule U of M such that $N \supset U$.

Proof: The proof runs as above. □

Definition 2.5[9]: Let $x \neq 0, x \in M$. Then x is said to be a minimal element if $< x > \supset P$,

P is a submodule of M, then either $P = < 0 >$ or $P = < x >$.

Note 2.6: If M_1 is a submodule of M, M_2 is a sub module of M_1 then M_2 need not be a sub-module of M.

Note 2.7: If M satisfies descending chain condition on $K-$subgroups, then M contains a submodule N of M such that $M \supset N$ and N is minimal and $N \neq < 0 >$.

For $x \in N, x \neq < 0 >$, $< x >$ is minimal.

Thus minimal elements exists with descending chain condition on $K-$subgroups. In fact, we can say that $M^{'}$ is any sub module of M, then there exists $x \in M^{'}, x \neq 0$ which is minimal in M.

All modules satisfy the condition (P).

Theorem 2.8: If M satisfies descending chain condition of $N-$subgroups and also Goldie condition, then there exists minimal elements $x_1, x_2, \ldots\ldots, x_n$ in M such that $< x_1 > \oplus < x_2 > \oplus \cdots$

$\cdots \oplus < x_n >$ is direct and M is an essential extension of $< x_1 > \oplus < x_2 > \oplus \cdots \oplus < x_n >$.

Proof: By descending chain condition there exists $x_1 \in M$ such that $< x_1 >$ is minimal.

If M is not an essential extension of $< x_1 >$, then there exists an $K-$subgroup N such that $< x_1 > \cap N = < 0 >$.

Therefore $< x_1 > \cap < N > = 0$.

So, descending chain condition on $K-$subgroups, $x_2 \in < N >$ which is minimal in M and $< x_1 > \oplus < x_2 >$ is direct.

By Goldie condition, after a finite number of steps we get minimal elements $x_1, x_2, \ldots\ldots, x_n$ such that $< x_1 > \oplus < x_2 > \oplus \cdots \oplus < x_n >$ is direct and $< x_1 > \oplus < x_2 > \oplus \cdots \oplus < x_n > \leq_e M$. □

Theorem 2.9: Let $x_1, x_2, \ldots \cdots, x_n$ and $y_1, y_2, \ldots \cdots, y_m$ be two sets of minimal elements such that

$< x_1 > \oplus < x_2 > \oplus \cdots \oplus < x_n > \leq_e M$ and $< y_1 > \oplus < y_2 > \oplus \cdots \oplus < y_m > \leq_e M$, then $n = m$.

Proof: Assume that $n < m$. Then $< x_1 > \oplus < x_2 > \oplus \cdots \oplus < x_n > \cap < y_i > \neq < 0 >$.

But $< y_i >$ is minimal implies that $< y_i > \subset < x_1 > \oplus < x_2 > \oplus \cdots \oplus < x_n > = S$

$\Rightarrow < y_1 > \oplus < y_2 > \oplus \cdots \oplus < y_m > \subset S$.

Similarly, $< x_1 > \oplus < x_2 > \oplus \cdots \oplus < x_n > \subset < y_1 > \oplus < y_2 > \oplus \cdots \oplus < y_m >$.

Therefore $S = < x_1 > \oplus < x_2 > \oplus \cdots \oplus < x_n > = < y_1 > \oplus < y_2 > \oplus \cdots \oplus < y_m > = S$.

Now $x_1 \in S$

$\Rightarrow \quad x_1 = z_1 + z_2 + \cdots + z_m , \quad z_i \in < y_i > \quad ; i = 1,2, \cdots, m.$

We can assume that $z_1 \neq 0$.

Then $x_1 - z_1 \in < y_2 > \oplus \cdots \oplus < y_m >$.

Therefore $z_1 = x_1 - (x_1 - z_1) \in < x_1 > \oplus < y_2 > \oplus \cdots\cdots \oplus < y_m >$.

Since $z_1 \neq 0$, $< z_1 > = < y_1 >$;

$\Rightarrow < y_1 > \subset < x_1 > \oplus < y_2 > \oplus \cdots \oplus < y_m >$.

Therefore, $S = < y_1 > \oplus \cdots \oplus < y_m > \subset < x_1 > \oplus < y_2 > \oplus \cdots \oplus < y_m >$.

Again $x_2 \in S = < x_1 > \oplus < y_2 > \oplus \cdots \oplus < y_m >$,

$\Rightarrow x_2 = t_1 + t_2 + \cdots + t_m , t_i \in < y_i > \quad ; i = 1,2, \ldots m$.

We can assume that $t_2 \neq 0$.

Then $t_1 + t_2 = t_2 + t_1^{'} , t_1^{'} \in < x_1 >$ as $< x_1 >$ is normal.

Therefore, $x_2 = t_2 + t_1^{'} + t_3 + \cdots + t_m$

$\Rightarrow - t_2 + x_2 = t_1^{'} + t_3 + \cdots + t_m$

$\Rightarrow -t_2 + x_2 \in < x_1 > \oplus < y_3 > \oplus \cdots \oplus < y_m >$

$\Rightarrow \quad t_2 \in < x_1 > \oplus < x_2 > \oplus < y_3 > \oplus \cdots \oplus < y_m >$.

Therefore $t_2 \neq 0, t_2 \in < y_2 >$ and $< y_2 >$ is minimal.

$\Rightarrow < y_2 > \subset < x_1 > \oplus < x_2 > \oplus < y_3 > \oplus \cdots \oplus < y_m >$

$\Rightarrow < x_1 > \oplus < y_3 > \oplus \cdots \oplus < y_m > \subset < x_1 > \oplus < x_2 > \oplus < y_3 > \oplus \cdots \oplus < y_m >$

$\Rightarrow S = < x_1 > \oplus < y_2 > \oplus \cdots \oplus < y_m > \subset < x_1 > \oplus < x_2 > \oplus < y_3 > \oplus \cdots \oplus < y_m >$.

Therefore, $S = < x_1 > \oplus < x_2 > \oplus < y_3 > \oplus \cdots \oplus < y_m >$.

Proceeding like this, we have $S = \langle x_1 \rangle \oplus \cdots \oplus \langle x_n \rangle \oplus \langle y_{n+1} \rangle \oplus \cdots \oplus \langle y_m \rangle$.
This cannot be.
Therefore $n \not< m$.
Hence, $n \geq m$.
Similarly, $m \geq n$.
Therefore $m = n$. □

Thus, if M satisfies descending chain condition of N subgroups, Goldie condition and property (P), then there exists minimal elements $x_1, x_2, \ldots, x_n$ in M such that $\langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots\cdots$
$\cdots \oplus \langle x_n \rangle \leq_e M$ and n depends only on M but not on the choice of minimal elements.

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