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Homomorphisms, Ideals and Congruence Relations of Pre A*-Algebra

Y. Praroopa[1] and J.V. Rao[2]

[1]*Asst. Professor in Mathematics,*
Andhra Loyola Institute of Engineering & Technology,
Vijayawada, A.P., India
E-mail: roopa169@rediffmail.com
[2]*Professor in Mathematics, Makelle University, Ithiopia.*
E-mail: drjvenkateswararao@gmail.com

Abstract

This paper is a study on algebraic structure of Pre A^*-algebra. We define Pre A^* – homomorphism, mono, epi, isomorphisms, automorphim of Pre A^*-algebra. We define ideal and Kernel of a Pre A^*-algebra. We prove Kernel is an ideal in Pre A^*- algebra. We prove some theorems on homomorphisms of Pre A*-algebras. Next we define congruence relation on Pre A*-algebra and we prove theorems on Pre A*-homomorphism ad congunce relation on Pre A*-algebra. We prove some propositions on Pre A* - algebra and Prove f(A) $\cong$ A/Kerf where f is a Pre A^* – homomorphism, f(A) is its image ad Kerf is its Kernel.

Keywords: Pre A^*-algebra, Pre A^* homomorphism, ideal, congruence relation on Pre A* - Algebra, Kernel) of Pre A* - algebra.

Introduction
Boolean algebras, essentially introduced by Boole in 1850's to codify the laws of thought, have been a popular topic of research since then. A major breakthrough was the duality of Boolean algebras and Boolean spaces as discovered by Stone in 1930's. Stone also proved that Boolean algebras and Boolean rings are essentially the same in the sense that one can convert via terms from one to the other. Since every Boolean algebra can be represented as a field of sets, the class of Boolean algebras is sometimes regarded as being rather uncomplicated. However, when one starts to look at basic questions concerning decidability, rigidity, direct products etc., they are associated with some of the most challenging results.

The study lattice theory had been made by Birkhoff (1948). In a draft paper, the equational theory of disjoint alternatives, Manes (1989) introduced the concept of Ada, (A, $\wedge$, $\vee$, (-)', $(-)^{\pi}$, 0, 1, 2) which however differs from the definition of the Ada by Manes (1993), While the Ada of the earlier draft seems to be based on extending the If –Then –Else concept more on the basis of Boolean algebras, the later concept is based on C- algebra

(A, $\wedge$, $\vee$, $(-)^{\sim}$) introduced by Fernando and Craig (1990).

Koteswara Rao (1994) firstly introduced the concept of A* - algebra (A, $\wedge$, $\vee$,*,$(-)^{\sim}$, $(-)_{\pi}$, 0, 1, 2) and studied the equivalnence with Ada by Manes (1989), C- algebra by Fernando and Craig (1990) and Ada by Manes (1993)) and its connection with 3-ring, stone type representation and introduced the concept of A* -clone and the If-Then-else structure over A*-algebra and ideal of A*-algebra. Venkateswara Rao (2000) introduced the concept Pre A* - algebra (A, $\wedge$, $\vee$, $(-)^{\sim}$) analogous to C-algebra as a reduct of A*- algebra. Recently Pre A* - algebra had been studied by Chandrasekhara Rao (2007) and Srinivasa Rao (2009)

Definition

An algebra (A,$\wedge$, $\vee$,$(-)^{\sim}$) where A is non-empty set with 1,

$\wedge,\vee$ are binary operations and $(-)^{\sim}$ is a unary operation satisfying

(a) $x^{\sim\sim} = x$, $\forall$, $x \in A$

(b) $x \wedge x = x, \quad \forall x \in A$

(c) $x \wedge y = y \wedge x, \quad \forall x, y \in A$

(d) $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}, \forall\, x, y \in A$

(e) $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \quad \forall x, y, z \in A$

(f) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \quad \forall x, y, z \in A$

(g) $x \wedge y = x \wedge (x^{\sim} \vee y), \forall x, y, z \in A$.

is called a Pre A*-algebra

Example

3 = {0, 1, 2} with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	$x^{\sim}$
0	1
1	0
2	2

Note
The elements 0,1,2 in the above example satisfy the following laws:

(a) $2^{\sim} = 2$

(b) $1 \wedge x = x$ for all $x \in 3$

(c) $0 \vee x = x$, for all $x \in 3$

(d) $2 \wedge x = 2 \vee x = 2$, $\forall x \in 3$.

Example: $2=\{0,1\}$ with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

x	$x^{\sim}$
0	1
1	0

Note

(i) $(2, \vee, \wedge, (-)^{\sim})$ is a Boolean algebra. So every Boolean algebra is a Pre A* algebra

(ii) The identities 1.1 (a) and 1.1 (d) imply that the varieties of Pre A* - algebras satisfies all the dual statements of 1.1 (a) to 1.1 (g).

Note
If (mn) is an axiom in Pre A* - algebra, then $(mn)^{\sim}$ is its dual.

§ 2. Homomorphisms, Ideals and Congruence Relations of Pre A*-Algebra.

Pre A* - Homomorphism

Definition

Let $\left(A_1, \wedge, \vee, (-)^{\sim}\right)$ and $\left(A_2, \vee, \wedge, (-)^{\sim}\right)$ be two Pre A^*-algebras. A mapping $f: A_1 \to A_2$ is called a Pre A^*-homomorphism if it satisfies the following

i. $f(a \wedge b) = f(a) \wedge f(b)$, $\forall a,b \in A$

ii. $f(a \vee b) = f(a) \vee f(b)$, $\forall a,b \in A$

iii. $f(a^{\sim}) = (f(a))^{\sim}$, $\forall a \in A$

Note
The general “morphism” concept has four different (though related) interpretations in

Pre A^*-algebras, each of which has important applications. These will now be described.

The homomorphism $f: A_1 \to A_2$ is onto, then f is called epimorphism

The homomorphism $f: A_1 \to A_2$ is one – one then f is called homomorphism

The homomorphism $f: A_1 \to A_2$ is one – one and onto then f is called an isomorphism and A_1, A_2 are isomorphic, denoted in symbols $A_1 \cong A_2$

An isomorphism from a PreA*-algebra A_1 into itself is called an automorphism

Definition

Let A_1, A_2 be two Pre A^*-algebras and $f: A_1 \to A_2$ be a homomorphism then the set

$\{x \in A_1 / f(x) = 0\}$ is called the Kernel of f and it is denoted by Kerf.

Example

Let A be a Pre A^*-algebra with 1, 0.Suppose that for every $x \in A - \{0,1\}$, $x \vee x \sim \neq 1$. Define $f: A \to \{0,1,2\}$ by f(1) = 1, f(0) = 0 and f(x) = 2 if $x \neq 0,1$. Then f is a Pre A^*-homomorphism.

Ideal in Pre A^*-algebra

Definition: An ideal is a non-void subset I of a Pre A^*-algebra A with the properties.

i. $a \in I, x \in A, x \leq a \Rightarrow x \in I$ (i.e., $a \wedge x \in I$)

ii. $a \in I, b \in I \Rightarrow a \vee b \in I$

Theorem

Under any Pre A^*-homomorphism f of a Pre A^*- algebra A onto a Pre A^* algebra B with 0, the set kerf (kernel of f) is an ideal in A.

Proof: Let $f: A \to B$ be a Pre A^*-homomorphism

Then Kerf = $\{x \in A / f(x) = 0\}$

If f(a) = 0 ad f(b) = 0

Then $f(a \vee b) = f(a) \vee f(b)$ = 0 V 0 = 0

$\Rightarrow a \vee b \in$ Kerf

i.e., if $a \in$ Kerf, $b \in$ Kerf $\Rightarrow a \vee b \in$ Kerf

$a \in$ Kerf $\Rightarrow f(a) = 0$

for $x \in A$, consider $f(a \wedge x) = f(a) \wedge f(x)$

$= 0 \wedge f(x)$

= 0

$\Rightarrow a \wedge x \in \text{Kerf}$

Therefore from (i) & (ii), Kerf is an ideal in A

Theorems on Pre A*-homomorphism

Theorem: Let $f: A \to B$ be a Pre A*-homomorphism from a Pre A*-algebra A into a Pre A*-algebra B ad Kerf = $\{x \in A / f(x) = 0\}$ is the Kernel then Kerf = $\{0\}$ if and only if f is one – one.

Proof: Suppose Kerf = $\{0\}$

To show that f is one-one:

For any $x, y \in A$, consider $f(x) = f(y)$

$\Rightarrow f(x) - f(y) = 0$

$\Rightarrow f(x - y) = 0$

$\Rightarrow x - y \in \text{Kerf} = \{0\}$

$\Rightarrow x - y = 0$

$\Rightarrow x = y$, $\forall x, y \in A$

Therefore f is one – one

Converse: suppose that f is one – one

$\Rightarrow f(x) = f(y) \Rightarrow x = y, \forall x, y \in A$

To show that Kert f = {0}

Let $x \in Kerf$

$\Leftrightarrow f(x) = 0$

$\Leftrightarrow x = 0$ (Since f is one – one)

Therefore Kerf = {0}

Lemma

Let $f: A_1 \to A_2$ be Pre A*-homomorphism where A_1, A_2 are Pre A*-algebras with 1_1 and 1_2-Then

If A_1 has the element 2, then f(2) is the element of A_2.

If $a \in B(A_1)$, then $f(a) \in B(A_2)$

where $B(A_1) = \{x \in A_1 / x \vee x^{\sim} = 1\}$

$B(A_2) = \{x \in A_2 / x \vee x^{\sim} = 1\}$

Note

If $f: A \to B$ and $g: B \to C$ are Pre A*-homomorphisms. So their composition or product gof : $A \to C$, which is defined by gof (a) = g(f(a) is also Pre A*-homomorphism.

Proposition

If $f : A \to B$ and $g : B \to C$ are Pre A*-homomorphisms. Then

i. If f and g are mono, so is gof
ii. If f and g are epi, so is gof
iii. If gof is mono, so is f
iv. If gof is epi, so is g

Proof: If $f : A \to B$ and $g : B \to C$ are Pre A*-homomorphisms. Then $gof : A \to C$ is also a Pre A*-homomorphisms.

i. Suppose f, g are one-one

Now suppose (gof)(a_1) = (gof)(a_2)

$\Rightarrow g(f(a_1)) = g(f(a_2))$

$\Rightarrow f(a_1) = f(a_2)$ (Since g is one –one)

$\Rightarrow a_1 = a_2$ (Since f is one-one)

Therefore gof is mono

ii. Suppose f, g are onto

Let $c \in C$, since g is onto, there exists $b \in B$ such that g(b) = c.

Since $b \in B$ and f is onto there exists $a \in A$ such that f(a) = b.

Therefore, g(b) = g(f(a)) = c

= (gof)(a) = c

Hence, for $c \in C$, there exists $a \in A$ such that (gof)(a) = c

This is true for every $c \in C$

Therefore gof is onto

Therefore gof is epimorphism.

iii. Suppose gof is mono

i.e., gof is one-one

We have to show f is one-one

Suppose f(a_1) = f(a_2)

$\Rightarrow g(f(a_1)) = g(f(a_2))$

$\Rightarrow (gof)(a_1) = (gof)(a_2)$

$\Rightarrow a_1 = a_2$ (Since gof is one-one)

Therefore f is one-one.

Hence f is mono.

Suppose gof is epi $\Rightarrow$gof is onto

We have to show g is onto

Since gof : $A \to C$ is onto, for any $c \in C$, there exist $a \in A$ such that

(gof)(a) = c

$\Rightarrow$g(f(a)) = c where f(a) $\in$ B

$\Rightarrow$f(a) = b, where $b \in B$

Therefore g(b) = c, for some $b \in B$

Therefore for $c \in C$, $\exists b \in B \ni$ g(b) = c ,

this is true for all $c \in C$

Hence g is onto

Thus g is epimorphism.

Corollary: The Pre A^* homomorphisms $f: A \to B$ is an isomorphism, if and only if, there exists a Pre A^*-homomorphism $g: B \to C$ such that fog is an automorphism of B and gof is an automorphism of A.

Proof: Suppose that the Pre A^*-homomorphism $f: A \to B$ is an isomorphism.
i.e., f is a bijection

Then $f^{-1}: B \to A$ is a bijection such that $fof^{-1} = I_B$

and $f^{-1}of = I_{A.}$

hence $f^{-1}: B \to A$ is a mapping such that $fof^{-1} = I_B$, which is an automoptism of B.

and $f^{-1}of = I_A$, which is an automorphism of A

Now we show that f^{-1} is a Pre A^*-homomorphism :

Since f is Pre A^*-homomorphism, we have f(1) = 1

$\Rightarrow f^{-1}(1) = 1$

Let b_1, $b_2 \in B$.

Since $f: A \to B$ is isomorphism, we have f is onto

Then $\exists$ a_1, a_2 $\in A \ni f(a_1) = b_1$, $f(a_2) = b_2$.

Therefore $f(a_1 v a_2) = f(a_1)$ v $f(a_2)$

$= b_1 \vee b_2$ (Since f is Pre A^*-homomorphism)

$\Rightarrow a_1 \vee a_2 = f^{-1}(b_1 \vee b_2)$

$\Rightarrow f^{-1}(b_1 \vee b_2) = f^{-1}(b_1) \vee f^{-1}(b_2)$

and $f(a_1 \wedge a_2) = f(a_1) \wedge f(a_2)$

$= b_1 \vee b_2$

$\Rightarrow a_1 \wedge a_2 = f^{-1}(b_1 \wedge b_2)$

$\Rightarrow f^{-1}(b_1 \wedge b_2) = f^{-1}(b_1) \wedge f^{-1}(b_2)$

Since

$$f^{-1}\left(b_1^{\sim}\right) = \left(f^{-1}(b_1)\right)^{\sim}$$

Therefore f^{-1} is a Pre A* - homomorphism

By taking g = f^{-1} we have $g: B \to A$ is a Pre A^*-homo, such that fog & gof are automorphisms of B & A respectively.

Converse

Conversely, assume that $g: B \to A$ is a Pre A^*-homomorphism such that fog & gof are auto of B and A respectively, where $f: A \to B$ is a Pre A^*-homomorphism

Since $fog: B \to B$ is an auto we have fog is an epimorphism.

$\Rightarrow$ f is epi (by proposition 2.11)

$\Rightarrow$ f is onto

Since gof is auto, we have gof is mono

$\Rightarrow$ f is mono (Since by proposition 2.11)

$\Rightarrow$ f is one – one

Therefore $f: A \to B$ is an isomorphism.

Congruence relation on Pre A* - algebra

Definition

A relation θ on a Pre A^*-algebra $(A, \wedge, \vee, (-)^{\sim})$ is called congruence relation if

(i) θ is an equivalence relation.

(ii) θ is closed under $\wedge, \vee, (-)^{\sim}$.

Lemma: Let $(A, \wedge, \vee, (-)^{\sim})$ be a Pre A*-algebra and let $a \in A$. Then the relation $\theta_a = \{(x, y) \in A \times A /\ a \wedge x = a \wedge y\}$ is a congruence relation

Proof: Since $a \wedge x = a \wedge x$, $x\ \theta_a x$

Hence the relation is reflexive.

If $x\ \theta_a y$ then $a \wedge x = a \wedge y$

$\Rightarrow a \wedge y = a \wedge x$

$\Rightarrow y\theta_a x$

Therefore the relation is symmetric

If $x\ \theta_a y$ and $y\theta_a z$ then

$a \wedge x = a \wedge y$ and $a \wedge y = a \wedge z$

Therefore $a \wedge x = a \wedge z$

$\Rightarrow x\theta_a z$

Therefore the relation is transitive.

Hence the relation is equivalence relation.

If $x, y, s, t \in A$ satisfy $x\ \theta_a y$, $s\ \theta_a t$ then $a \wedge x = a \wedge y$, $a \wedge s = a \wedge t$

Now $a \wedge (x \vee s) = (a \wedge x) \vee (a \wedge s)$

$= (a \wedge y) \vee (a \wedge t)$

$= a \wedge (y \vee t)$.

This shows that $(x \vee s)\ \theta_a (y \vee t)$.

Therefore $x\ \theta_a y$, $s\ \theta_a t$ then $(x \vee s)\ \theta_a (y \vee t)$.

Hence θ_a is closed under $\vee$.

If $x, y, s, t \in A$ satisfy $x\ \theta_a y$, $s\ \theta_a t$ then $a \wedge x = a \wedge y$, $a \wedge s = a \wedge t$

Now $a \wedge (x \wedge s) = (a \wedge x) \wedge s$

$= (a \wedge y) \wedge s$

$= a \wedge (y \vee s)$

$= a \wedge (s \vee y)$

$= (a \wedge s) \wedge y$

$= (a \wedge t) \wedge y$

$= a \wedge (t \wedge y)$

$= a \wedge (y \wedge t)$.

This shows that $(x \wedge s)\ \theta_a (y \wedge t)$.

Therefore $x\ \theta_a y$, $s\theta_a t$ then $(x \wedge s)\ \theta_a (y \wedge t)$.

Hence θ_a is closed under $\wedge$.

If $x, y \in A$ satisfy $x\ \theta_a y$, then $a \wedge x = a \wedge y$

$\Rightarrow a^{\sim} \vee x^{\sim} = a^{\sim} \vee y^{\sim}$

$\Rightarrow a \wedge (a^{\sim} \vee x^{\sim}) = a \wedge (a^{\sim} \vee y^{\sim})$

$\Rightarrow a \wedge x^{\sim} = a \wedge y^{\sim}$

$\Rightarrow x^{\sim}\ \theta_a y^{\sim}$

Therefore θ_a is closed under $(-)^{\sim}$

Therefore θ_a is a congruence relation

Theorem: Let A be a Pre A* algebra with 1 and $x \in B(A)$, then following are equivalent

(a) $x \in B(A)$

(b) $\psi_x = \theta_{x^{\sim}}$

(c) ψ_x is congruence relation on A

(d) ψ_x is reflexive on A

(e) ψ_x is symmetric on A

Proof:

$a \Rightarrow b$, suppose $x \in B(A)$, then by lemma 3.7, we have $\Gamma_x(p, q) = p$ if and only if $x^{\sim} \wedge p = x^{\sim} \wedge q$, for all $p, q \in A$. This shows that $\psi_x = \theta_{x^{\sim}}$

$b \Rightarrow c$ is clear

$c \Rightarrow d$ is clear

$d \Rightarrow e$ suppose that ψ_x is reflexive on A, $p, q \in A$ and $p, q \in \psi_x$

then $p, q \in \theta_{x^{\sim}}$, so that $x^{\sim} \wedge p = x^{\sim} \wedge q$

Now $\Gamma_x(q, p) = (x \wedge q) \vee (x^{\sim} \wedge \text{p})$

$= (x \wedge q) \vee (x^{\sim} \wedge q)$

$= \Gamma_x(q, q) = q$ (since ψ_x is reflexive on A)

Therefore $(q, p) \in \psi_x$ and hence ψ_x is symmetric on A

$e \Rightarrow a$ suppose that ψ_x is symmetric on A

$\Gamma_x(x \vee x^{\sim}, 1) = (x \wedge (x \vee x^{\sim})) \vee (x^{\sim} \wedge 1)$

$= x \vee (x^{\sim} \wedge 1)$

$= x \vee x^{\sim}$

$\therefore (x \vee x^{\sim}, 1) \in \psi_x$ and hence $(1, x \vee x^{\sim}) \in \psi_x$

$\Rightarrow 1 = \Gamma_x(1, x \vee x^{\sim})$

$= (x \wedge 1) \vee (x^{\sim} \wedge (x \vee x^{\sim}))$

$= x \vee x^{\sim}$

This implies that $x \in B(A)$.

Note: If θ_x is a reflexive relation on a Pre A^*-algebra with 1 and $x \in B(A)$ then θ_x is symmetric and transitive, hence a congruence relation.

Theorem
Let A be a Pre A^*-algebra with 1 and $x \in A$, then the mapping

$\alpha_x : A \to A_x$ defined by

$\alpha_x(s) = x \wedge s$ for all $s \in A$ is a homomorphism of A onto A_x with Kernel θ_x and hence

$A/\theta_x \cong A_x$

Proof
Define $\alpha_x : A \to A_x$ by $\alpha_x(s) = x \wedge s, \forall s \in A$

Where $A_x = \{x \wedge s / s \in A\}$

For $s \in A$, $x \wedge (x \wedge s) = x \wedge s$, $x \wedge s \leq x$

Hence $x \wedge s \in A_x$

Let $s, t \in A_x$

Then $\alpha_x(s \wedge t) = x \wedge s \wedge t$

$= x \wedge s \wedge x \wedge t$

$= \alpha_x(s) \wedge \alpha_x(t)$

$\alpha_x(s^{\sim}) = x \wedge s^{\sim}$

$= x \wedge (x\sim \vee s\sim)$

$= x \wedge (x \wedge s)\sim\ = (\alpha_x(s))^{\sim}$

Therefore $\alpha_x(s\sim) = (\alpha_x(s))^{\sim}$

We can prove that $\alpha_x(s \vee t) = \alpha_x(s) \vee \alpha_x(t)$

Hence α_x is a Pre A^*- homomorphism

Now $s \in A_x$, we have $\alpha_x(s) = s$.

Therefore α_x is onto homomorphism.

Hence by the fundamental theorem of homomorphism.

$A/_{Ker\alpha_x} \cong A_x$ and

Ker $\alpha_x = \{(s,t) \in AxA / \alpha_x(s) = \alpha_x(t)\}$

$= \{(s,t) \in AxA / x \wedge s = x \wedge t\}$

$= \theta_x$

Thus $A/\theta_x \cong A_x$

Proposition

If $f : A \to B$ is a Pre A^y-homomorphism then $\exists$ a congruence relation θ_x on A, an epimorphism $\alpha_x : A \to A_x$ and a homomorphism $\alpha : A_x \to B$. such that f = $\alpha o \alpha_x$

Proof: Suppose $f : A \to B$ is a Pre A^*-homomorphism.

Define a relation θ_x ={(s,t) $\in$ A x A / x $\wedge$ s = x $\wedge$ t}

Then by lemma 3.2, θ_x is a congruence relation on A.

Define a mapping $\alpha_x : A \to A_x$ by α_x (s) = x $\wedge$ s $\forall$ s $\in$ A

Where A_x = {x $\wedge$ s/s $\in$ A}

By theorem 3.5, α_x is a Pre A^*-homomorphism which is onto.

Hence α_x is epimorphism from A onto A_x.

Define $\alpha : A_x \to B$ by $\alpha(A_x) = f(s),\ \forall s \in A$

i.e., $\alpha(x \wedge s) = f(s),\ \forall s \in A$

Cleanly α is well – defined and one – one

(Since $\alpha(x \wedge s) = \alpha(y \wedge t)$)

$\Leftrightarrow x \wedge s = y \wedge t$ (by def.))

(i) $\alpha((x \wedge s) \vee (x \wedge t)) = \alpha(x \wedge (s \vee t))$

$= f(s \vee t)$

$= f(s) \vee f(t)$ (Since f is Pre A^*-homo)

$= \alpha(x \wedge s) \vee \alpha(x \wedge t)$

(ii) Similarly we can show that

$\alpha((x \wedge s) \wedge \alpha(x \wedge t)) = \alpha(x \wedge s) \wedge \alpha(x \wedge t)$

(iii) $\alpha(A_x^{\sim}) = \alpha(x \wedge (s)^{\sim})$

$= \alpha(x^{\sim} \vee s^{\sim})$

$= \alpha(x \wedge s)^{\sim}$

$= (\alpha(A_x))^{\sim}$

Therefore $\alpha(A_x^{\sim}) = (\alpha(A_x))^{\sim}$

Therefore α is a Pre A^*-homomorphism hence α is mono.

For any $a \in A$, $\alpha o \alpha_x(a) = \alpha(\alpha_x(a))$

$= \alpha(x \wedge a)$

$= f(a)$

Therefore $\alpha o \alpha_x = f, \ \forall a \in A$

Proposition

If f is a Pre A^*- homomorphism of a Pre A^*-algebra A into another Pre A*-algebra, then $f(A) \cong A/f^{-1}(0)$

Where f(A) is called the image, $f^{-1}(0) = \{a \in A/f(a) = 0\}$ the Kernel of f.

Proof: $f : A \to B$ is a Pre Ay-homomorphism.

Then by Proposition 3.6 there exists a congruence relation θ_x on A, an epimorphism $\alpha_x : A \to A_x$ and a monomorphism $\alpha : A_x \to B$ such that $\alpha o \alpha_x = f$

$\Rightarrow \alpha o \alpha_x(a) = f(a), \forall a \in A$

$\Rightarrow f(a) = \alpha o \alpha_x(a)$

$= \alpha(\alpha_x(a))$

$\cong \alpha_x(a)$ (Since α is mono)

= A_x (Since α_x is onto)

$\therefore f(a) \cong A_x, \forall a \in A$

Hence $f(a) \cong A_x \to (1)$

Since $\alpha_x : A \to A_x$ is onto then by fundamental theorem of homomorphism we have

$$A_x \cong \frac{A}{Ker\alpha_x}$$

Ker α_x = {(s,t) $\in$ A x A / α_x(s) = α_x(t)}

= {(s,t) $\in$ A x A / $x \wedge s = x \wedge t$ }

= θ_x

$\therefore A/\theta_x \cong A_x$

$\therefore A/Ker\alpha_x \cong A_x$

Since Ker α_x = Ker f;

(Verification : Let s $\in$ Ker α_x

$\Leftrightarrow \alpha_x(s) = 0$

$\Leftrightarrow x \wedge s = 0$

$\Leftrightarrow \alpha(x \wedge s) = 0$ (Since α isone-one)

$\Leftrightarrow f(s) = 0$ (Since $\alpha : A_x \to B$ by $\alpha(x \wedge s) = f(s), \forall s \in A$)

$\Leftrightarrow s \in Kerf$)

Hence A/ Kerf $\cong$ $A_x \cong$ f(A) (by (1))

Therefore f(A) $\cong$ A/Kerf

References

[1] Birkhoff G (1948). Lattice Theroy, American Mathematical Society, Colloquium publishers, New York.

[2] Chandrasekhararao K, Venkateswararao J, Koteswararao P (2007). Pre A* - Algebras, J. Instit, Mathe. Comput. Sci. Math. Ser. 20(3) :157-164

[3] Fernando G, Craig C-Squir (1990). The algebra of conditional logic, Algebra Universalis 27: 88-10.

[4] Koteswararao P (1994). A*-algebras and if-then-else structures (doctoral) thesis, Acharya Nagarjuna University, A.P., India.

[5] Venkateswararao J (2000). On A*-algebras (doctoral thesis), Acharya Nagarjuna University, A.P., India.
[6] Srinivasa Rao K. (2009) on Pre A*- algebras (doctoral thesis), Acharya Nagarjuna University, A.P., India.
[7] Venkateswara Rao. J and Srinivasa Rao. K, Pre A*-Algebra as a Poset, African Journal Mathematics and Computer Science Research.Vol.2 (4), pp 073-080, May 2009.
[8] Howie. M.H: An introduction to Semigroup Theory, Academic press,1976
[9] Jacobson N: (1984) Basic algebra II, Hindustan Publishing Corporation (India), Delhi
[10] Lambek. J: Lectures on rings and modules, Chelsea Publishing Company, New York(1986)
[11] Manes E.G: The Equational Theory of Disjoint Alternatives, personal communication to Prof. N.V. Subrahmanyam (1989)
[12] Manes E.G: Ada and the Equational Theory of If-Then-Else, Algebra Universalis 30(1993), 373-394.
[13] Neal HerryMc Coy: (1948) Rings and Ideals, The Mathematical assertions of America.
[14] Venkateswara Rao. J and Srinivasa Rao. K, Congruence relation on Pre A*-Algebra, Journal of Mathematical Sciences, accepted (Appear in Aug/Nov 2009)
[15] Venkateswara Rao and Srinivasa Rao. K, Pre A*-Algebra as a Poset, Africa Journal Mathematics and Computer Science Research.Vol.2 (4), pp 073-080, May 2009.