Common Fixed Point for Compatible Mappings on a 2-Metric Space

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Abstract

The aim of the present paper is to obtain i) a common fixed point theorem for compatible mappings by using the concept of asymptotic regularity and ii) a common fixed point theorem using the concept of joint reciprocal continuity in 2-metric spaces. A supporting example is also given.

Keywords: common fixed point, Compatible mappings and asymptotic regularity.

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Introduction

Gähler[2] introduced the concept of 2-metric space as a natural generalization of a metric space. Some fixed point theorems in 2-metric spaces are obtained in Iseki[3],Rhoades[5] and Jungck[4]. Gähler[2] introduced the notions of reciprocal continuity and asymptotic regularity for a pair of self maps on a 2-metric space. Using the above concepts Badshah and Gopal Meena[1] proved a fixed point theorem for a pair of self-maps on a 2-metric space using asymptotic regularity.

The present paper is a generalization of a result of Badshah and Gopal
Meena([1], Theorem 2).

In this paper, we obtain i) a common fixed point theorem (Th. (2.1)) for compatible mappings by using the concept of asymptotic regularity and ii) a common fixed point theorem (Th. (2.6)) using the concept of joint reciprocal continuity. We start with some definitions.

**Definition 1.1 (Gähler[2]):** Let $X$ be a non-empty set with real valued function $d: X^2 \to \mathbb{R}$ satisfying:

For two distinct points $x, y, z \in X$ such that $d(x, y, z) \neq 0$

- $d(x, y, z) = 0$ only if at least two of $x, y$ and $z$ are equal
- $d(x, y, z) = d(x, z, y) = d(y, z, x)$ and
- $d(x, y, z) \leq d(x, y, w) + d(w, u, z) + d(u, y, z)$ for all $x, y, z, u \in X$.

The function $d$ is called a 2-metric on $X$ and the pair $(X, d)$ is called a 2-metric space.

**Definition 1.2 (Gähler[2]):** Let $(X, d)$ be a 2-metric space.

A sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ if

$$\lim_{n \to \infty} d(x, x_n, x) = 0, \forall x \in X.$$

A sequence $\{x_n\}$ is said to be a Cauchy sequence in $X$ if

$$\lim_{n \to \infty} d(x_n, x_m, x) = 0, \forall x \in X.$$

A 2-metric space $(X, d)$ is said to be complete if every Cauchy sequence in $X$ converges to a point in $X$.

Note: i) In a 2-metric space $(X, d), d: X^3 \to \mathbb{R}$ is continuous if

$$x_n \to x, \quad y_n \to y \quad \text{implies} \quad d(x, y, z) \to d(x, y, z), \forall x \in X \text{ for } n \to \infty.$$

ii) If $x_n \to x$ and $x_n \to y$ as $n \to \infty$ then $x = y$.

**Definition 1.3 (Gähler[2]):** For self-mappings, $S$ and $T$ of a 2-metric space $(X, d)$, the pair $(S, T)$ is called reciprocally continuous if

$$\lim_{n \to \infty} d(STx_n, Sx_n, a) = 0 = d(TSx_n, Tx_n, a), \forall a \in X, \text{ whenever } \{x_n\} \text{ is a sequence in } X \text{ such that } \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = x, \text{ for some } x \in X.$$

**Definition 1.4 (Gähler[2]):** For self-mappings, $S$ and $T$ of a 2-metric space $(X, d)$, a sequence $\{x_n\}$ in $X$ is called asymptotically regular with respect to the pair $(S, T)$ if

$$\lim_{n \to \infty} d(Sx_n, Tx_n, a) = 0, \forall a \in X.$$

The pair $(S, T)$ is called compatible if

$$\lim_{n \to \infty} d(STx_n, TSx_n, a) = 0, \forall a \in X,$$

whenever $\{x_n\}$ is a sequence in...
such that 

\[ \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = x, \text{ for some } x \in X. \]

**Definition 1.5:** Suppose \( P, S, T \) are self maps on a 2-metric space \((X, d)\). The pair \((S, T)\) is said to be joint reciprocal continuous with respect to \( P \) if there exists a sequence \( \{v_n\} \) in \( X \) such that 

\[ \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n \quad \text{and} \quad \lim_{n \to \infty} d(Px_n, Sx_n, a) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(PTx_n, TPx_n, a) = 0 \quad \forall a \in X. \]

Note: If \( \{v_n\} \) is a sequence in \( X \) such that 

\[ \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n \quad \text{and} \quad \{P, S\} \quad \text{and} \quad \{P, T\} \]

are compatible then \((S, T)\) is jointly reciprocally continuous with respect to \( P \).

**Notation:** \( \Phi = \{\phi : [0, \infty) \to [0, \infty), \phi \text{ is continuous and } \phi(t) < t, \forall t > 0\} \).

### Main Results

We begin with our first main result, from this we obtain corollaries of the main result and finally show that the result of Badshah and Gopal Meena[1] follows as a corollary.

**Theorem 2.1:** Let \( P, S, T \) be self-mappings of a complete 2-metric space \((X, d)\) satisfying:

\[ \forall x, y, a \in X \quad \text{and for some } \varphi \in \Phi, \]

**i)** the pair \( (P, S) \) and \( (P, T) \) are compatible

**ii)** there exists a sequence \( \{v_n\} \) which is asymptotically regular with respect \( \{P, S\} \) and \( \{P, T\} \)

\[ d \text{ is continuous.} \]

Then \( P, S, T \) have a unique common fixed point in \( X \).

**Proof:** Let \( \{v_n\} \) be a sequence in \( X \) satisfying condition **iii**.

By taking \( x = x_n \) and \( y = x_m \) in (i), we obtain

\[ d(Py, Sx_n, a) \leq \varphi \left( \max \left( d(Py, Ty, a), d(Py, Sx_n, a), d(Py, Ty, a), d(Py, Sx_n, a) \right) \right) \]

\[ = \varphi \left( \max \left( d(Py, Ty, a) + d(Py, Sx_n, a) + d(Py, Ty, a) + d(Py, Sx_n, a) \right) \right) \]

(2.1.1)

On letting \( n \to \infty \) and using condition **iii**, we get

\[ \lim_{n \to \infty} d(Px_n, Px_m, a) = 0 \quad \forall a \in X. \]

This implies that the sequence \( \{Px_n\} \) is a Cauchy sequence in \( X \), since \( X \) is complete with respect to 2-metric \( d \).

This implies
Now
\[ d(Sx_n, z, a) \leq d(Sx_n, z, Px_n) + d(Sx_n, Px_n, a) + d(Px_n, z, a) \]
On letting \( n \to \infty \), using condition \((ii)\) and equation \((2.1.2)\), we get
\[ \lim_{n \to \infty} d(Sx_n, z, a) = 0 \quad \forall a \in X. \]
This implies \( Sx_n \to z \) in \( X \).
Similarly we can obtain
\[ Tx_n \to z \text{ in } X. \] (2.1.3)
Since we can write
\[ d(PSx_n, Sz, a) \leq d(PSx_n, Sz, SPx_n) + d(SPx_n, SPx_n, a) + d(SPx_n, Sz, a) \]
On letting \( n \to \infty \), using the conditions \((i), (iv)\) and \((v)\), we get
\[ \lim_{n \to \infty} d(PSx_n, Sz, a) = 0 \quad \forall a \in X, \]
\( (\forall P x_n \to z \text{ and } S \text{ is continuous implies } SPx_n \to Sz) \)
This implies \( PSx_n \to z \).
(2.1.4)
Similarly we can prove \( PTx_n \to Tx \).
By taking \( x = Sx_n \) and \( y = Tx_n \) in \((i)\), we obtain
\[ d(PSx_n, y, PTx_n, a) \leq \varphi(\max\{d(PSx_n, y, PTx_n, a), d(PSx_n, z, a)\}) \]
\[ d(PSx_n, y, a) \leq d(PSx_n, z, a) \]
\[ d(PSx_n, y, a) \leq d(PSx_n, z, a) \]
\[ d(PSx_n, y, a) \leq d(PSx_n, z, a) \]
\[ \lim_{n \to \infty} d(PSx_n, Sz, a) = 0 \quad \forall a \in X, \]
\( (\forall P x_n \to z \text{ and } S \text{ is continuous implies } SPx_n \to Sz) \)
This implies \( PSx_n \to z \).
(2.1.5)
From condition \((iv)\) we have \( S \) and \( T \) are continuous, applying continuity of \( S \) and \( T \) in \((2.1.3)\) and \((2.1.4)\), we get
\[ SSx_n \to Sz, TTx_n \to Tz \text{ and } STx_n \to Sz. \] (2.1.6)
On letting \( n \to \infty \) in \((2.1.7)\), using \((2.1.5),(2.1.6),(2.1.8)\) and condition \((v)\), we get
\[ d(Tz, Sz, a) = 0 \quad \forall a \in X. \]
This implies \( Tz = Sz. \) (2.1.9)
Similarly by taking \( x = Tx_n \) and \( y = z \) in \((i)\), we get
\[ d(PTx_n, z, a) \leq \varphi(\max\{d(PTx_n, z, a), d(PTx_n, Tx_n, a), d(Tx_n, z, a)\}) \]
\[ d(PTx_n, z, a) \leq \varphi(\max\{d(PTx_n, z, a), d(PTx_n, Tx_n, a), d(Tx_n, z, a)\}) \]
\[ d(PTx_n, z, a) \leq \varphi(\max\{d(PTx_n, z, a), d(PTx_n, Tx_n, a), d(Tx_n, z, a)\}) \]
On letting \( n \to \infty \), using \((2.1.7),(2.1.8),(2.1.9)\) and condition \((v)\), we get
\[ d(Tz, Pz, a) = 0 \quad \forall a \in X. \]
This implies \( Tz = Pz. \) (2.1.10)
Therefore \( Sz = Tz = Pz. \) (2.1.11)
By taking \( x = x_n \) and \( y = z \) in \((i)\), we get
\[ d(x_n, Pz, a) \leq \varphi(\max\{d(Pz, Tz, a), d(Pz, x_n, a)\}) \]
\[ d(x_n, Pz, a) \leq \varphi(\max\{d(Pz, Tz, a), d(Pz, x_n, a)\}) \]
\[ d(x_n, Pz, a) \leq \varphi(\max\{d(Pz, Tz, a), d(Pz, x_n, a)\}) \]
On letting \( n \to \infty \) and using \((2.1.2),(2.1.3),(2.1.11)\) and condition \((iv)\), we get
\[ d(x, Pz, a) = 0 \quad \forall a \in X. \]
This implies \( z \) is a fixed point of \( P \).
Therefore \( z \) is a fixed point of \( S, T \) and \( P \) in \( X. \)
Suppose if \( x \) is common fixed point of \( S, T \) and \( P \) in \( X. \) Then it can be easily proved that \( x = z. \) Hence \( z \) is a unique common fixed point of \( S, T \) and \( P \) in \( X. \)
Now we have the following corollaries of theorem 2.1.
**Corollary 2.2:** Let \( P, S \) and \( T \) be self-mappings of a complete 2-metric space \((X, d)\) satisfying:

1. \( d(Fx, Py, a) \leq \lambda \max \{ (d(Fy, Ty, a)(1 + d(Fx, Sx, a)), d(Fx, Sx, a)(1 + d(Fy, Ty, a)), d(Fx, Sy, a)] \}

\( \forall x, y, a \in X, \lambda \in [0, 1) \) and also satisfying condition \((iv), (v), (vi), (vii)\) and \((viii)\) of theorem 2.1.

Then \( P, S \) and \( T \) have a unique common fixed point in \( X \).

**Corollary 2.3:** Let \( P, S \) and \( T \) be self-mappings of a complete 2-metric space \((X, d)\) satisfying:

1. \( d(Fx, Py, a) \leq \alpha \max \{ (d(Fy, Ty, a)(1 + d(Fx, Sx, a)), d(Fx, Sx, a)(1 + d(Fy, Ty, a)), d(Fx, Sy, a)] \}

\( \forall x, y, a \in X, \alpha \in [0, 1) \) and \( \beta = \alpha + \beta \) are non-negative numbers such that \( \alpha + \beta < 1 \) and also satisfying condition \((ii), (iii), (iv)\) and \((v)\) of theorem 2.1.

Then \( P, S \) and \( T \) have a unique common fixed point in \( X \).

**Corollary 2.4:** Let \( P, S \) and \( T \) be self-mappings of a complete 2-metric space \((X, d)\) satisfying:

1. \( d(Fx, Py, a) \leq \alpha \max \{ (d(Fy, Ty, a)(1 + d(Fx, Sx, a)), d(Fx, Sx, a)(1 + d(Fy, Ty, a)), d(Fx, Sy, a)] \}

\( \forall x, y, a \in X, \alpha \in [0, 1) \) and \( \beta = \alpha + \beta \) are non-negative numbers such that \( \alpha + \beta < 1 \) and also satisfying condition \((ii), (iii), (iv)\) and \((v)\) of theorem 2.1.

Then \( P, S \) and \( T \) have a unique common fixed point in \( X \).

The following result due to Badshah and Gopal Meena[1] is a corollary of the above result.

**Corollary 2.5 (Badshah and Gopal Meena[1],Theorem 2):** Let \( P, S \) and \( T \) be self-mappings of a complete 2-metric space \((X, d)\) satisfying:

1. \( d(Fx, Py, a) \leq \alpha \max \{ (d(Fy, Ty, a)(1 + d(Fx, Sx, a)), d(Fx, Sx, a)(1 + d(Fy, Ty, a)), d(Fx, Sy, a)] \}

\( \forall x, y, a \in X, \alpha \in [0, 1) \) and \( \beta = \alpha + \beta \) are non-negative numbers such that \( \alpha + \beta < 1 \), and also satisfying condition \((ii), (iii), (iv)\) and \((v)\) of theorem 2.1.

Then \( P, S \) and \( T \) have a unique common fixed point in \( X \).

Now we state our second main result which uses the concept of joint reciprocal continuity.

**Theorem 2.6:** Let \( P, S \) and \( T \) be self-mappings of a complete 2-metric space \((X, d)\) satisfying:

\[ \max \{ (d(Fy, Ty, a)(1 + d(Fx, Sx, a)), d(Fx, Sx, a)(1 + d(Fy, Ty, a)), d(Fx, Sy, a)] \}

\( \forall x, y, a \in X \) and \( 0 < \lambda < 1 \)

1. \( S \) and \( T \) are continuous
2. \( d \) is continuous
3. \((S, T)\) is joint reciprocal continuous w.r.t. \( P \) in \( X \).

Then \( P, S \) and \( T \) have a unique common fixed point in \( X \).
Proof: From condition (iv) there exists a sequence \( \{x_n\} \) in \( X \) such that
\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Px_n = \lim_{n \to \infty} Tx_n = z \quad \text{for some} \quad z \in X
\] (2.5.1)
and
\[
\lim_{n \to \infty} d(PSx_n, SPx_n, a) = 0 = \lim_{n \to \infty} d(PTx_n, TPx_n, a) \quad \forall a \in X
\] (2.5.2)
Applying condition (ii) in equation (2.5.1) and using this in the equation (2.5.2), we get
\[
PSx_n \to Sz \quad \text{and} \quad PTx_n \to Tz.
\] (2.5.3)
By taking \( x = Sx_n, y = Tx_n \) in (i) and letting \( n \to \infty \), using conditions (ii), (iii) and (2.5.2) and (2.5.3), we get \( Sz = Tz \). (as derived in theorem 2.1)
Similarly taking \( x = Tx_n \) and \( y = z \) in (i) and letting \( n \to \infty \), we get \( Pz = Tz \) (as derived in theorem 2.1).
Taking \( x = x_n \) and \( y = z \) in (i) and letting \( n \to \infty \), we get \( z = Pz \).
This is \( z \) is a fixed point of \( P \). Hence \( z \) is a common fixed point of \( S, T \) and \( P \) in \( X \). Suppose if \( x \) is common fixed point of \( S, T \) and \( P \) in \( X \). Then it can be easily proved that \( x = z \). Hence \( z \) is a unique common fixed point of \( S, T \) and \( P \) in \( X \).

Now we have the following corollaries of theorem 2.6.

**Corollary 2.7:** Let \( P, S \) and \( T \) be self-mappings of a complete 2-metric space \( (X, d) \) satisfying: condition (i) of corollary (2.2) and conditions (ii), (iii) and (iv) of theorem (2.6). Then \( P, S \) and \( T \) have a unique common fixed point in \( X \).

**Corollary 2.8:** Let \( P, S \) and \( T \) be self-mappings of a complete 2-metric space \( (X, d) \) satisfying: conditions (i) of corollary (2.3) and also conditions (ii), (iii) and (iv) of theorem (2.6). Then \( P, S \) and \( T \) have a unique common fixed point in \( X \).

**Corollary 2.9:** Let \( P, S \) and \( T \) be self-mappings of a complete 2-metric space \( (X, d) \) satisfying: conditions (i) of corollary (2.4) and also conditions (ii), (iii) and (iv) of theorem (2.6). Then \( P, S \) and \( T \) have a unique common fixed point in \( X \).

**Corollary 2.10:** Let \( P, S \) and \( T \) be self-mappings of a complete 2-metric space \( (X, d) \) satisfying: conditions (i) of corollary 2.5 and also the conditions (ii), (iii) and (iv) of theorem (2.6). Then \( P, S \) and \( T \) have a unique common fixed point in \( X \).
Example 2.11:
Let $X = \mathbb{R} \times \mathbb{R}$ for $A, B \in X$, denote the Euclidean distance $A$ and $B$ by $|A - B|$. Define $d: X^3 \rightarrow \mathbb{R}$ by $d(A, B, C) = \min\{|A - B|, |B - C|, |C - A|\}$. Then $(X, d)$ is a complete 2-metric space. Let $A_0 \in X$. Define the mappings $P, S$ and $T$ on $X$ as $P(A) = A_0$ and $S = T = I$. Define $\phi(t) = \eta t, 0 < \eta < 1$ then $\phi \in \Phi$. Then $P, S$ and $T$ satisfy all the properties of Theorem 2.1 and Theorem 2.6 and $A_0$ is the unique common fixed point of $P, S$ and $T$ in $X$.

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