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Common Fixed Point for Compatible Mappings on a 2-Metric Space

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Abstract

The aim of the present paper is to obtain i) a common fixed point theorem for compatible mappings by using the concept of asymptotic regularity and ii) a common fixed point theorem using the concept of joint reciprocal continuity in 2- metric spaces. A supporting example is also given.

Keywords: common fixed point, Compatible mappings and asymptotic regularity.

Mathematical subject Classification : 47 H 10,54H25

Introduction

Gāhler[2] introduced the concept of 2-metric space as a natural generalization of a metric space. Some fixed point theorems in 2-metric spaces are obtained in Iseki[3],Rhoades[5] and Jungck[4]. Gāhler[2] introduced the notions of reciprocal continuity and asymptotic regularity for a pair of self maps on a 2-metric space. Using the above concepts Badshah and Gopal Meena[1] proved a fixed point theorem for a pair of self-maps on a 2-metric space using asymptotic regularity.

The present paper is a generalization of a result of Badshah and Gopal

Meena([1],Theorem 2).

In this paper, we obtain i) a common fixed point theorem (Th. (2.1)) for compatible mappings by using the concept of asymptotic regularity and ii) a common fixed point theorem (Th. (2.6)) using the concept of joint reciprocal continuity. We start with some definitions.

Definition 1.1 (Gähler[2]): Let X be a non-empty set with real valued function $d: X^3 \to R$ satisfying :

For two distinct points $x, y \text{ in } X, \quad \text{there exists } z \text{ in } X \text{ such that } d(x,y,z) \neq 0$

$d(x,y,z) = 0$ only if at least two of $x, y \text{ and } z \text{ are equal}$

$d(x,y,z) = d(x,z,y) = d(y,z,x)$ and

$d(x,y,z) \leq d(x,y,u) + d(x,u,z) + d(u,y,z)$ for all $x, y, z, u \text{ in } X.$

The function d is called a 2-metric on X and the pair (X,d) is called a 2-metric space.

Definition 1.2 (Gähler[2]) : Let (X,d) be a 2-metric space .
A sequence $\{x_n\}$ is said to be convergent to a point
$x \text{ in } X \text{ if } \lim_{n\to\infty} d(x_n, x, a) = 0, \forall a \in X.$

A sequence $\{x_n\}$ is said to be a Cauchy sequence in X if $\lim_{n\to\infty} d(x_m, x_n, a) = 0, \forall a \in X$.

A 2-metric space (X,d) is said to be complete if every Cauchy sequence in X converges to a point in X .

Note: i) In a 2-metric space $(X,d), d: X^3 \to R$ is continuous if
$x_n \to x, \quad y_n \to y \text{ implies } d(x_n, y_n, a) \to d(x,y,a), \forall a \in X \text{ for } n \to \infty$.

ii) If $x_n \to x \text{ and } x_n \to y \text{ as } n \to \infty \text{ then } x = y.$

Definition 1.3 (Gähler[2]): For self-mappings, $S \text{ and } T$ of a 2-metric space (X,d), the pair (S,T) is called reciprocally continuous if $\lim_{n\to\infty} d(STx_n, Sx, a) = 0 = d(TSx_n, Tx, a) \forall a \in X$, whenever $\{x_n\}$ is a sequence in $X \text{ such that } \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x, \text{ for some } x \in X$.

Definition 1.4 (Gähler[2]): For self-mappings , $S \text{ and } T$ of a 2-metric space (X,d)
A sequence $\{x_n\}$ in X is called asymptotically regular with respect to the pair (S,T) if $\lim_{n\to\infty} d(Sx_n, Tx_n, a) = 0 \ \forall a \in X$.

The pair (S,T) is called compatible if
$\lim_{n\to\infty} d(STx_n, TSx_n, a) = 0 \ \forall a \in X$, whenever $\{x_n\}$ is a sequence in

X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x, for\ some\ x \in X$.

Definition 1.5: Suppose $P, S\ and\ T$ are self maps on a 2-metric space (X, d).The pair (S, T) is said to be joint reciprocal continuous with respect to P if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n\ and\ \lim_{n\to\infty} d(PSx_n, SPx_n, a) = 0\ and\ \lim_{n\to\infty} d(PTx_n, TPx_n, a) = 0\ \forall a \in X$.

Note : If $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n$ and $(P, S)\ and\ (P, T)$ are compatible then (S, T) is jointly reciprocally continuous with respect to P .

Notation: $\Phi = \{\varphi | \varphi: [0, \infty) \to [0, \infty), \phi \text{ is continous and } \phi(t) < t\ \ \forall t > 0\}$.

Main Results

We begin with our first main result, from this we obtain corollaries of the main result and finally show that the result of Badshah and Gopal Meena[1] follows as a corollary.

Theorem 2.1: Let $P, S\ and\ T$ be self-mappings of a complete 2-metric space (X, d) satisfying:

$i)\ d(Px, Py, a) \leq \varphi\ (\max\ [\{ d(Py, Ty, a)(1 + d(Px, Sx, a)), d(Px, Sx, a)(1 + d(Py, Ty, a), d(Px, Sx, a), d(Py, Sy, a)\}])$

$\forall x, y, a \in X\ and\ \ for\ some\ \varphi \in \Phi,$

$ii)\ the\ pair\ (P, S)\ \ and\ (P, T)\ \ are\ compatible$

$iii)$ there exists a sequence $\{x_n\}$ which is asymptotically regular with respect $(P, S) and\ (P, T)$

$iv)\ \ S\ and\ T\ are\ continous\ \ v)\ \ d\ is\ continous.$

Then $P, S\ and\ T$ have a unique common fixed point in X .

Proof: Let $\{x_n\}$ be a sequence in X satisfying $condition\ (iii)$.

By taking $x = x_n\ and\ y = x_m\ \ in\ (i),$ we obtain

$$[d(Px]_n, Px_m, a) \leq \varphi\ [(\max]\left\{ d(Px_m, Tx_m, a)\left(1 + d(Px_n, Sx_n, a)\right),\ \ d(Px_n, Sx_n, a)\left(1 + (Px_m, Tx_m, a)\right), d(Px_n, Sx_n, a), d(Px_m, Sx_m, a)\right\}\Big)\Big) \tag{2.1.1}$$

On letting $n \to \infty$ and using condition (iii), we get

$\lim_{n\to\infty} d(Px_n, Px_m, a) = 0\ \forall a \in X$.

This implies that the sequence $\{Px_n\}$ is a Cauchy sequence in X , since X is complete with respect to 2-metric $d.$

This implies

$Px_n \to z$, $\quad$ *for some* $z \epsilon X$. (2.1.2)

Now

$d(Sx_n, z, a) \leq d(Sx_n, z, Px_n) + d(Sx_n, Px_n, a) + d(Px_n, z, a)$

On letting $n \to \infty$, using condition (iii) and equation (2.1.2), we get

$\lim_{n\to\infty} d(Sx_n, z, a) = 0 \ \forall a \in X.$

This implies $Sx_n \to z \text{ in } X.$ (2.1.3)

Similarly we can obtain

$Tx_n \to z \text{ in } X.$ (2.1.4)

Since we can write

$d(PSx_n, Sz, a) \leq \quad d(PSx_n, Sz, SPx_n) + d(PSx_n, SPx_n, a) + d(SPx_n, Sz, a)$

On letting $n \to \infty$, using the conditions $(ii), (iv) and (v),$ we get

$\lim_{n\to\infty} d(PSx_n, Sz, a) = 0 \ \forall a \in X,$

$(\because Px_n \to z \text{ and } S \text{ is continuous implies } SPx_n \to Sz)$

This implies $PSx_n \to Sz$. (2.1.5)

Similarly we can prove $PTx_n \to Tz$. (2.1.6)

By taking $x = Sx_n \text{ and } y = Tx_n$ in (i), we obtain

$d(PSx_n, PTx_n, a) \leq \varphi(max\{d(PTx_n, TTx_n, a)(1 + d(PSx_n, SSx_n, a)),$
$d(PSx_n, SSx_n, a)(1 + d(PTx_n, TTx_n, a)),$
$d(PSx_n, SSx_n, a), d(PTx_n, STx_n, a)\})$ (2.1.7)

From condition (iv) we have $S \text{ and } T$ are continuous, applying continuity of $S \text{ and } T$ in (2.1.3) and (2.1.4) ,we get

$SSx_n \to Sz, TTx_n \to Tz \text{ and } STx_n \to Sz$ (2.1.8)

On letting $n \to \infty$ in (2.1.7), using (2.1.5),(2.1.6),(2.1.8) and condition (v), we get

$d(Tz, Sz, a) = 0, \forall a \in X .$

$\text{This implies } Tz = Sz.$ (2.1.9)

Similarly by taking $x = Tx_n \text{ and } y = z \text{ in } (i)$, we get

$d(PTx_n, Pz, a) \leq \varphi(\max\{d(Pz, Tz, a)(1 + d(PTx_n, STx_n, a), d(PTx_n, STx_n, a)(1 + d(Pz, Tz, a) , d(PTx_n, STx_n, a), d(Pz, Sz, a)\})$

On letting $n \to \infty$, using ((2.1.7),(2.1.8),(2.1.9) and condition(v), we get

$d(Tz, Pz, a) = 0, \forall a \in X .$

$\text{This implies } Tz = Pz.$ (2.1.10)

Therefore $Sz = Tz = Pz.$ (2.1.11)

By taking $x = x_n \text{ and } y = z \text{ in } (i)$, we get

$d(Px_n, Pz, a) \leq \varphi(\max\{d(Pz, Tz, c)(1 + d(Px_n, Sx_n, a), \quad d(Px_n, Sx_n, a)(1 + d(Pz, Tz, a) , d(Px_n, Sx_n, a), d(Pz, Tz, a)\})$

On letting $n \to \infty$ and using (2.1.2),(2.1.3),(2.1.11) and condition(iv), we get

$d(z, Pz, a) = 0, \forall a \in X$. $\text{This implies } Pz = z.$

Therefore $z \text{ is a fixed point of } P.$ Hence z is a common fixed point of $S, T \text{ and } P \text{ in } X.$

Suppose if x is common fixed point of $S, T \text{ and } P \text{ in } X.$ Then it can be easily proved that $x = z$. Hence z is a unique common fixed point of $S, T \text{ and } P \text{ in } X.$

Now we have the following corollaries of theorem 2.1.

Corollary 2.2 : Let $P, S \; and \; T$ be self-mappings of a complete 2-metric space (X, d) satisfying:

$i)\; d(Px, Py, a) \le \lambda \max [\{ d(Py, Ty, a)(1 + d(Px, Sx, a)), \quad d(Px, Sx, a)(1 + d(Py, Ty, a), d(Px, Sx, a), d(Py, Sy, a)\}]$

$\forall x, y, a \in X, 0 < \lambda < 1$ and also satisfying condition $(ii), (iii), (iv) and (v)$ of theorem 2.1.

Then $P, S \; and \; T$ have a unique common fixed point in X .

Corollary 2.3: Let $P, S \; and \; T$ be self-mappings of a complete 2-metric space (X, d) satisfying:

$i)\; d(Px, Py, a) \le \alpha \max [\{ d(Py, Ty, a)(1 + d(Px, Sx, a)), \quad d(Px, Sx, a)(1 + d(Py, Ty, a)\} + \beta \max\{d(Px, Sx, a), d(Py, Sy, a)\}]$

$\forall x, y, a \in X, \alpha \; and \; \beta \; are \; non-negative \; numbers \; such \; that \; \alpha + \beta < 1 \; and$ also satisfying condition $(ii), (iii), (iv) and (v)$ of theorem 2.1.

Then $P, S \; and \; T$ have a unique common fixed point in X .

Corollary 2.4: Let $P, S \; and \; T$ be self-mappings of a complete 2-metric space (X, d) satisfying:

$i)\; d(Px, Py, a) \le \alpha \; [\, d(Py, Ty, a)(1 + d(Px, Sx, a)) +] \; \beta\{d(Px, Sx, a) + d(Py, Sy, a)\}$

$\forall x, y, a \in X, \alpha \; and \; \beta \; are \; non-negative \; numbers \; such \; that \; \alpha + \beta < 1$ and also satisfying condition $(ii), (iii), (iv) and (v)$ of theorem 2.1.

Then $P, S \; and \; T$ have a unique common fixed point in X .

The following result due to Badshah and Gopal Meena[1] is a corollary of the above result.

Corollary 2.5 (Badshah and Gopal Meena[1],Theorem 2): Let $P, S \; and \; T$ be self-mappings of a complete 2-metric space (X, d) satisfying:

$i)\; d(Px, Py, a) \le \alpha \, d(Py, Ty, a)(1 + d(Px, Sx, a))/(1 + d(Sx, Ty, a))$
$+\beta\{d(Px, Sx, a) + d(Py, Sy, a)\}$

$\forall x, y, a \in X, \alpha \; and \; \beta \; are \; non-negative \; numbers \; such \; that \; \alpha + \beta < 1,$

and also satisfying condition $(ii), (iii), (iv) and (v)$ of theorem 2.1.

Then $P, S \; and \; T$ have a unique common fixed point in X .

Now we state our second main result which uses the concept of joint reciprocal continuity.

Theorem 2.6: Let $P, S \; and \; T$ be self-mappings of a complete 2-metric space (X, d) satisfying:

$\max[[\{ d(Py, Ty, a)(1 + d(Px, Sx, a)), \quad d(Px, Sx, a)(1 + d(Py, Ty, a), d(Px, Sx, a), d(Py, Sy, a)\}]])$

$\forall x, y, a \in X \; and \; 0 < \lambda < 1$

$ii)\; S \; and \; T \; are \; continous$

$iii)\; d \; is \; continous.$

$iv)\; (S, T) \; is \; joint \; reciprocal \; continous \; w.r.t. \; P \; in \; X$.

Then $P, S \; and \; T$ have a unique common fixed point in X .

Proof: From $condition(iv)$ there exists a sequence $\{x_n\}$ $in\ X\ such\ that$

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} Tx_n = z\ for\ some\ z \in X \tag{2.5.1}$$

and

$$\lim_{n\to\infty} d(PSx_n, SPx_n, a) = 0 = \lim_{n\to\infty} d(PTx_n, TPx_n, a) \forall a \in X \tag{2.5.2}$$

Applying $condition\ (ii)$ in equation (2.5.1) and using this in the equation (2.5.2),we get

$$PSx_n \to Sz\ and\ PTx_n \to Tz\ . \tag{2.5.3}$$

By taking $x = Sx_n, y = Tx_n$ in (i) and letting $n \to \infty$, using conditions (ii) , (iii) and (2.5.2) and (2.5.3), we get $Sz = Tz$. (as derived in theorem 2.1)

Similarly taking $x = Tx_n\ and\ y = z\ in\ (i)$ and letting $n \to \infty$, we get $Pz = Tz$ (as derived in theorem 2.1).Therefore $Sz = Pz = Tz$.

Taking $x = x_n\ and\ y = z\ in\ (i)$ and letting $n \to \infty$, we get $z = Pz$.

This is $z\ is\ a\ fixed\ point\ of\ P.$ Hence z is a common fixed point of $S, T\ and\ P\ in\ X.$ Suppose if x is common fixed point of $S, T\ and\ P\ in\ X.$ Then it can be easily proved that $x = z$. Hence z is a unique common fixed point of $S, T\ and\ P\ in\ X.$

Now we have the following corollaries of theorem 2.6.

Corollary 2.7: Let $P, S\ and\ T$ be self-mappings of a complete 2-metric space (X, d) satisfying: condition (i) of corollary (2.2) and conditions $(ii), (iii)\ and\ (iv)$ of theorem (2.6).

Then $P, S\ and\ T$ have a unique common fixed point in X .

Corollary 2.8: Let $P, S\ and\ T$ be self-mappings of a complete 2-metric space (X, d) satisfying: conditions (i) of corollary(2.3) and also conditions $(ii), (iii)\ and\ (iv)$ of theorem (2.6).

Then $P, S\ and\ T$ have a unique common fixed point in X .

Corollary 2.9: Let $P, S\ and\ T$ be self-mappings of a complete 2-metric space (X, d) satisfying: conditions (i) of corollary (2.4) and also conditions $(ii), (iii)\ and\ (iv)$ of theorem (2.6).

Then $P, S\ and\ T$ have a unique common fixed point in X .

Corollary 2.10: Let $P, S\ and\ T$ be self-mappings of a complete 2-metric space (X, d) satisfying:

conditions (i) of corollary 2.5 and also the conditions $(ii), (iii)\ and\ (iv)$ of theorem (2.6).

Then $P, S\ and\ T$ have a unique common fixed point in $X.$

Example 2.11:
Let $X = R \times R$ for $A, B \in X,$ denote the Euclidean distance A and B by $|A - B|$.

Define $d: X^3 \to R$ by $d(A, B, C) = \min\{|A - B|, |B - C|, |C - A|\}$. Then (X, d) is a complete 2-metric space. Let $A_0 \in X.$ Define the mappings P, S and T on X as $P(A) = A_0$ $\forall A \in X$ and $S = T = I.$
$Define$ $\varphi(t) = \eta\, t, 0 < \eta < 1$ $then$ $\varphi \in \Phi$. Then P, S and T satisfy all the properties of Theorem 2.1 and Theorem 2.6 and A_0 is the unique common fixed point of P, S and T in $X.$

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