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On Generalized Useful Inaccuracy Measure of Order α & Type β and 1:1 Coding

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Abstract

In this paper, we have established some noiseless coding theorem for a generalized parametric useful mean length. Further, lower bounds on exponentiated useful codeword length have been obtained in terms of the useful inaccuracy of order α and type β and the generalized average useful codeword length.

Keywords: Generalized inaccuracy measure, useful mean code word length, Holder's inequality.

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Introduction

Consider the following model for a finite random experiment S :

$$S_n = [E; P; U] \tag{1.1}$$

where $E = (E_1, E_2, .., E_n)$ is a finite system of events $P = (p_1, p_2, ..., p_n)$, $0 \le p_i \le 1, \sum p_i = 1,$ is the probability distribution and $U = (u_1, u_2, ..., u_n)$, $u_i \ge 0, i = 1,2,..., n$ is the utility distribution. The u_i's are non-negative real numbers.

Now let us suppose that experimenter asserts that the ith outcome E_i has the probability q_i whereas the true probability is p_i, with $\sum p_i = \sum q_i = 1$. Thus we have two utility information schemes, (1.1) of a set of n events after an experiment and

$$S_n^* = [E; Q; U],\ 0 \le q_i \le 1,\ \sum q_i = 1,\ u_i \ge 0, \tag{1.2}$$

Unless otherwise stated $\sum$ will stand for $\sum_{i=1}^{n}$ and the logarithms are to the base D (D>1) throughout the paper.

The quantitative-qualitative measure of inaccuracy [6,23] associated with the statement of and experimenter is given by

$$I(P;Q;U) = -\sum u_i p_i \log q_i \tag{1.3}$$

By considering weighted mean codeword length [10]

$$L(U) = \frac{\sum u_i p_i l_i}{\sum u_i p_i} \tag{1.4}$$

Taneja and Tuteja [23] derived lower and upper bounds on $L(U)$ in terms of $I(P;Q;U)$ .Longo [16], Gurdial and Pessoa [11], Autar and Khan [3], Hooda and Bhakar[12], Bhatia [4], and Singh, Kumar and Tuteja [22] considered the problem of 'useful' information measures and used it studying the noiseless coding theorems of source involving utilities.

In section 2, we have established a generalized coding theorem for personal probability codes by considering useful inaccuracy of order α and type β .

The mean length of a noiseless uniquely decodable code for a discrete random variable X satisfies

$$H(X) \le L_{UD} < H(X) + 1 \tag{1.5}$$

where

$$H(X) = -\sum p_i \log p_i \tag{1.6}$$

is the Shannon's entropy [20] of the random variable X . Shannon's restriction of coding of X to prefix codes is highly justified by the implicit assumption that the description will be concatenated and thus must be uniquely decodable. Since the set of allowed codeword lengths is the same for the uniquely decodable and instantaneous codes, cf. [1], [2], the expected codeword length is the same for both the set of codes.

There are some communication situations in which a random variable X is being transmitted rather than a sequence of random variables. For this context Leung-Yang-Cheong and Cover [15] considered one to one codes i.e., codes which assign a distinct binary code to each outcome of the random variable X without regard to the condition that concatenations of the descriptions must be uniquely decipherable.

Bhatia [5], [7], have extended the idea of the one to one code to the Kerridge's inaccuracy[13] and also derived lower bounds to the exponentiated mean codeword length for the best one to one codes in terms of a generalized inaccuracy of order α .

In section 3, we generalized the idea of the best 1:1 code to useful inaccuracy of order α and type β .

Coding Theorem

Consider a function

$$I_{\alpha}^{\beta}(P;Q;U)=\frac{1}{1-\alpha}\log\left[\frac{\sum u_i^{\beta}p_i^{\beta}q_i^{\beta(\alpha-1)}}{\sum u_i^{\beta}p_i^{\beta}}\right],\ \alpha>0(\neq 1),\beta>0. \tag{2.1}$$

When $\beta=1$, (2.1) reduces to a measure of useful information measure of order α due to Bhatia [5].

When $\beta=1$, $u_i=1$ for each $i=1,2,...,n$, (2.1) reduces to the inaccu-racy measure given by Nath [17], further it reduces to Renyi's [12] entropy by taking $p_i=q_i$ for each $i=1,2,...,n$.

When $\beta=1$, $u_i=1$ for each $i=1,2,...,n$ and $\alpha\to 1$, (2.1) reduces to the measure due to Kerridge [13].

When $u_i=1$ for each $i=1,2,...,n$ and $p_i=q_i$, for each $i=1,2,...,n$ the measure (2.1) becomes the entropy for the β- power distribution derived from P studied by Roy[19].

We call (2.1) the generalized useful inaccuracy measure of order α and type β.

Further consider

$$L_{\beta}^{t}(U)=\frac{1}{t}\log\left[\sum p_i^{\beta}\left(\frac{u_i^{\beta}}{\sum u_i^{\beta}p_i^{\beta}}\right)^{t+1}D^{tl_i}\right],\ -1<t<\infty \tag{2.2}$$

For $\beta=1$, $L_{\beta}^{t}(U)$ in (2.2) reduces to the useful mean length $L^{t}(U)$ of the code given by Bhatia [5].

For $\beta=1$, $u_i=1$ for each $i=1,2,...,n$, $L_{\beta}^{t}(U)$ in (2.2) reduces to the mean length given by Campbell [8].

For $\beta=1$, $u_i=1$ for each $i=1,2,...,n$ and $\alpha\to 1$, $L_{\beta}^{t}(U)$ in (2.2) reduces to the optimal code length identical to Shannon [20].

For $u_i=1$ for each $i=1,2,...,n$, $L_{\beta}^{t}(U)$ in (2.2) reduces to the mean length given by Khan and Haseen [14].

Now we find the lower bounds of $L_{\beta}^{t}(U)$ in terms of $I_{\alpha}^{\beta}(P;Q;U)$ under the condition

$$\sum p_i^{\beta}q_i^{-\beta}D^{-l_i}\leq 1 \tag{2.3}$$

where D is the size of the code alphabet. Inequality (2.3) is a generalization of Kraft's inequality. A code satisfying generalized Kraft's inequality (2.3) would be termed as personal probability code.

Theorem 2.1: Let $\{u_i\}_{i=1}^n$, $\{p_i\}_{i=1}^n$, $\{q_i\}_{i=1}^n$ and $\{l_i\}_{i=1}^n$ satisfy the condition (2.3), then

$$L_\beta^t(U) \geq I_\alpha^\beta(P;Q;U) \tag{2.4}$$

where $\alpha = \frac{1}{t+1}$, the equality occurs if and only if

$$l_i = -\log \frac{u_i^\beta q_i^{\alpha\beta}}{\sum u_i^\beta p_i^\beta q_i^{\beta(\alpha-1)}} \tag{2.5}$$

Proof: By Holder's inequality [21]

$$\sum x_i y_i \geq \left(\sum x_i^p\right)^{1/p} \left(\sum y_i^q\right)^{1/q} \tag{2.6}$$

For all $x_i, y_i > 0, i = 1,2.....n$ and $\frac{1}{p} + \frac{1}{q} = 1,\ p < 1(\neq 0), q < 0$ or $q < 1(\neq 0), p < 0.$

We see that equality holds if and only if there exists a positive constant c such that

$$x_i^p = c y_i^q \tag{2.7}$$

Making the substitutions $p = -t$, $q = \frac{t}{1+t}$

$$x_i = p_i^{\frac{-\beta}{t}} \left(\frac{u_i^\beta}{\sum u_i^\beta p_i^\beta}\right)^{-\left(\frac{1+t}{t}\right)} D^{-l_i}\ ,\ y_i = p_i^{\beta\left(\frac{1+t}{t}\right)} \left(\frac{u_i^\beta}{\sum u_i^\beta p_i^\beta}\right)^{\frac{1+t}{t}} q_i^{-\beta}$$

in (2.6) and using (2.3), we get

$$\left[\sum p_i^\beta \left(\frac{u_i^\beta}{\sum u_i^\beta p_i^\beta}\right)^{t+1} D^{l_i t}\right]^{\frac{1}{t}} \geq \left[\frac{\sum u_i^\beta p_i^\beta q_i^{\beta(\alpha-1)}}{\sum u_i^\beta p_i^\beta}\right]^{\frac{1+t}{t}}$$

Taking logarithms of both the sides with base D, we obtain (2.4).

Next, we obtain a result giving an upper bound to the generalized average 'useful' codeword length.

Theorem 2.2: By property choosing the length $l_1, l_2, ..., l_n$ in the code of Theorem 2.1, $L_\beta^t(U)$ can be made to satisfy the inequality

$$L_\beta^t(U) < I_\alpha^\beta(P;Q;U) + 1 \tag{2.8}$$

Proof: From (2.5) it can be concluded that it is always possible to have a code satisfying

$$-\log\left(\frac{u_i^{\beta} q_i^{\alpha\beta}}{\sum u_i^{\beta} p_i^{\beta} q_i^{\beta(\alpha-1)}}\right) \le l_i < -\log\left(\frac{u_i^{\beta} q_i^{\alpha\beta}}{\sum u_i^{\beta} p_i^{\beta} q_i^{\beta(\alpha-1)}}\right) + 1$$

or

$$\left[\frac{u_i^{\beta} q_i^{\alpha\beta}}{\sum u_i^{\beta} p_i^{\beta} q_i^{\beta(\alpha-1)}}\right]^{-1} \le D^{l_i} < D\left[\frac{u_i^{\beta} q_i^{\alpha\beta}}{\sum u_i^{\beta} p_i^{\beta} q_i^{\beta(\alpha-1)}}\right]^{-1} \tag{2.9}$$

From (2.9), we have

$$D^{l_i} < D\left[\frac{u_i^{\beta} q_i^{\alpha\beta}}{\sum u_i^{\beta} p_i^{\beta} q_i^{\beta(\alpha-1)}}\right]^{-1} \tag{2.10}$$

Raising both sides of (2.10) to the power $\left(\frac{1-\alpha}{\alpha}\right)$, we get

$$D^{l_i\left(\frac{1-\alpha}{\alpha}\right)} < D^{\left(\frac{1-\alpha}{\alpha}\right)}\left[\frac{u_i^{\beta} q_i^{\alpha\beta}}{\sum u_i^{\beta} p_i^{\beta} q_i^{\beta(\alpha-1)}}\right]^{\left(\frac{\alpha-1}{\alpha}\right)} \tag{2.11}$$

Multiplying both sides of (2.11) by $p_i^{\beta}\left(\frac{u_i^{\beta}}{\sum u_i^{\beta} p_i^{\beta}}\right)^{\frac{1}{\alpha}}$, summing over i and after that raising to the power $\left(\frac{\alpha}{1-\alpha}\right)$, we get

$$\left[\sum p_i^{\beta}\left(\frac{u_i^{\beta}}{\sum u_i^{\beta} p_i^{\beta}}\right)^{\frac{1}{\alpha}} D^{l_i\left(\frac{1-\alpha}{\alpha}\right)}\right]^{\left(\frac{\alpha}{1-\alpha}\right)} < D\left[\frac{\sum u_i^{\beta} p_i^{\beta} q_i^{\beta(\alpha-1)}}{\sum u_i^{\beta} p_i^{\beta}}\right]^{\left(\frac{1}{1-\alpha}\right)}$$

Taking logarithms on both sides and using the relation $\alpha = \frac{1}{t+1}$, we get (2.8).

Lower bound on the exponentiated average 'useful' codeword length for the best 1:1 code

Let X be a random variable taking on a finite number of values $x_1, x_2, ..., x_n$ with probabilities $(p_1, p_2, ..., p_n)$ and utilities $(u_1, u_2, ..., u_n)$. Let $l_i, i = 1,2,...,n,$ be the lengths of the code words in the best 1:1 binary code (0, 1, 00, 10, 01, 11, 000,…), for encoding the random variable X, l_i is the length of the codeword assigned to the output x_i. It is clear that $l_1 \le l_2 \le l_3 \le ...$ and in general $l_i = \left\lceil \log_2\left(\frac{i+2}{2}\right)\right\rceil$, where $\lceil S \rceil$ denotes the smallest integer greater than or equal to S. Thus the average 'useful' codeword length for the best 1:1 code is given by

$$L_{1:1}(U) = \frac{\sum u_i p_i \left\lceil \log_2\left(\frac{i+2}{2}\right)\right\rceil}{\sum u_i p_i} \tag{3.1}$$

When utilities are ignored, (3.1) reduces to $L_{1:1}$, cf. [15].

From (2.2), the exponentiated average 'useful' codeword length for binary codes can be given by

$$L_\beta^t(U) = \frac{1}{t}\log_2\left[\sum p_i^\beta \left(\frac{u_i^\beta}{\sum u_i^\beta p_i^\beta}\right)^{t+1} 2^{tl_i}\right] \tag{3.2}$$

When $\beta = 1$, (3.2) reduces to Bhatia's [5] 'useful' codeword length for the best 1:1 code. Thus the exponentiated average 'useful' codeword length for the best 1:1 code is given by

$$L_{\beta,1:1}^t(U) = \frac{1}{t}\log_2\left[\sum p_i^\beta \left(\frac{u_i^\beta}{\sum u_i^\beta p_i^\beta}\right)^{t+1} 2^{t\left\lceil \log_2\left(\frac{i+2}{2}\right)\right\rceil}\right] \tag{3.3}$$

We will now prove the following theorem, which gives a lower bound on $L_{\beta,1:1}^t$.

Theorem 3.1: For $I_\alpha^\beta(P;Q;U)$, $L_\beta^t(U)$ and $L_{\beta,1:1}^t$ as given in (2.1), (3.2) and (3.3) respectively, the following estimates hold:

$$L_{\beta,1:1}^t(U) \ge I_\alpha^\beta(P;Q;U) - \log_2\left[\sum \frac{p_i^\beta}{q_i^\beta}\left(\frac{2}{i+2}\right)\right], \tag{3.4}$$

and

$$L_{\beta,1:1}^t(U) \ge L_\beta^t(U) - \log_2\left[\sum \frac{p_i^\beta}{q_i^\beta}\left(\frac{1}{i+2}\right)\right] - 2 \tag{3.5}$$

Proof From (3.3), we have

$$L^t_{\beta,1:1}(U) \geq \frac{1}{t}\log_2\left[\sum p_i^\beta \left(\frac{u_i^\beta}{\sum u_i^\beta p_i^\beta}\right)^{t+1}\left(\frac{i+2}{2}\right)^t\right] \tag{3.6}$$

Now

$$I_\alpha^\beta(P;Q;U) - L^t_{\beta,1:1}(U)$$

$$\leq \left(\frac{t+1}{t}\right)\log_2\left[\frac{\sum u_i^\beta p_i^\beta q_i^{-\beta\left(\frac{t}{t+1}\right)}}{\sum u_i^\beta p_i^\beta}\right] - \frac{1}{t}\log_2\left[\sum p_i^\beta \left(\frac{u_i^\beta}{\sum u_i^\beta p_i^\beta}\right)^{t+1}\left(\frac{i+2}{2}\right)^t\right]$$

$$= \log_2\left[\frac{\sum u_i^\beta p_i^\beta q_i^{-\beta\left(\frac{t}{t+1}\right)}}{\sum u_i^\beta p_i^\beta}\right]^{\left(\frac{t+1}{t}\right)}\left[\sum p_i^\beta \left(\frac{u_i^\beta}{\sum u_i^\beta p_i^\beta}\right)^{t+1}\left(\frac{i+2}{2}\right)^t\right]^{-\frac{1}{t}} \tag{3.7}$$

Applying Holder's inequality to (3.7), we obtain

$$I_\alpha^\beta(P;Q;U) - L^t_{\beta,1:1}(U) \leq \log_2\left[\sum \frac{p_i^\beta}{q_i^\beta}\left(\frac{2}{i+2}\right)\right]$$

which gives (3.4).

Now from (2.8)

$$L^t_\beta(U) < I_\alpha^\beta(P;Q;U) + 1$$

So

$$L^t_\beta(U) - L^t_{\beta,1:1}(U) < I_\alpha^\beta(P;Q;U) - L^t_{\beta,1:1}(U) + 1$$

$$\leq 2 + \log_2\left[\sum \frac{p_i^\beta}{q_i^\beta}\left(\frac{1}{i+2}\right)\right]$$

which proves (3.5).

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