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Cone Metric Spaces and Fixed Point Theorems of Generalized Contractive Mappings

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Abstract

We obtain sufficient conditions for the existence of points of coincidence and common fixed points of three self mappings satisfying a contractive type condition in cone metric spaces. Our results generalize several well-known recent results.

Introduction and Preliminaries

Guang and Zhang [3] recently introduced the concept of cone metric spaces and established some fixed point theorems for contractive type mappings in a normal cone metric space. Subsequently, some other authors [1, 4, 12] studiedthe existence of points of coincidence and common fixed points of mappings satisfying a contractive type condition in cone metric spaces. Afterwards, Rezapour and Hamlbarani [9] proved fixed point theorems of contractive type mappings by omitting the assumption of normality in cone metric spaces. In this paper we obtain points of coincidence and common fixed points for three self mappings satisfying Jungck [5] type contractive condition with the assumption of normality in cone metric spaces.

First we recall Jungck's [5] theorem

Theorem 1.1: Let (X, ρ) be a complete metric space. Let f be a continuous self-map on X and g be any self-map on X that commutes with f. Further let f and g satisfy $g(X) \subseteq f(X)$ and there exists a constant $\lambda \in (0, 1)$ such that for every $x, y \in X$, $\rho(gx, gy) \leq \lambda \rho(fx, fy)$.

Then f and g have a unique common fixed point.

Sessa [11] generalized the concept of commuting mappings by calling self mappings f, g on a metric space X, weakly commuting if and only if $d(fgx, gfx) \leq d(fx, gx)$ for all $x \in X$

Clearly commuting mappings are weakly commuting but converse is not true in general (see [11]).

Afterwards, many authors obtained nice fixed point theorems by using this concept.

Thus Jungck [6] and Pant [8] introduced some less restrictive concepts of compatible mappings and R-weakly commuting mappings respectively. Later on, it has been noticed that compatible mappings and R-weakly commuting mappings commute at their coincidence point. Jungck and Rhoades [7], then defined a pair (f, g) of self-mappings to be weakly compatible if they commute at their coincidence point (i.e. $fgx = gfx$ whenever $fx = gx$).

Definition 1.2: (L.G. Haung and X. Zhang [3]): Let E be a real Banach space and P a subset of E. P is called a cone if

(i) P is closed, non empty and $P \neq \{0\}$;

(ii) $ax + by \in P \; \forall x, y \in P$ and non negative real numbers a, b;

(iii) $P \cap (-P) = \{0\}$.

Note also that the relations $int\, P + int\, P \subseteq int\, P$ and $\lambda\, int\, P \subseteq int\, P$ $(\lambda > 0)$ hold. For a given cone $P \subseteq E$, we can define on E a partial ordering $\leq$ with respect to P by putting $x \leq y$ if and only if $y - x \in P$. Further, $x < y$ stands for $x \leq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in int\, P$, where $int\, P$ denotes the interior of P.

Definition 1.3: (L.G. Haung and X. Zhang [3])

Let E be a real Banach space and $P \subset E$ be a cone. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies

$$\| x \| \leq K \| y \|$$

The least positive number K satisfying the above inequality is called the normal constant of P.

In the following, we always suppose that E is a real Banach space, P is a cone in E and E is endowed with the partial ordering induced by P.

Definition 1.4: (L.G. Haung and X. Zhang [3])
Let X be a nonempty set. A mapping $d: X \times X \to E$ satisfying $0 \leq d(x,y)\ \forall\, x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$; $d(x,y) = d(y,x)\ \forall x,y \in X$; $d(x,y) \leq d(x,z) + d(z,y)\ \forall x,y,z \in X$, is called a cone metric on X, while (X,d) is called a cone metric space.

Definition 1.5: (L.G. Haung and X. Zhang [3])
Let *(X, d)* be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$. $\{x_n\}$ converges to x if for every $c \in E$ with $0 << c$, there is an n_o such that for all $n \geq n_o$, $d(x_n, x) << c$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$

(ii) If for any $c \in P$ with $0 << c$, there is an n_o such that for all $n, m \geq n_o$, $d(x_n, x_m) << c$, then $\{x_n\}$ is called a Cauchy sequence in X.

(iii) (X,d) is called a complete cone metric space, if every Cauchy sequence in X is convergent in X.

Definition 1.6 (M. Abbas [1]): Let f and T be self maps of a nonempty set X. If there exists $x \in X$ such that $fx = Tx$ then x is called a coincidence point of f and T, while $y = fx = Tx$ is called a point of coincidence of f and T.

Main Results

In this section, we prove two main results and obtain the results of [2] as corollaries.

We start with a lemma, which will be required in the sequel.

Lemma 2.1 (Azam, Arshad and Beg [2]): Let X be a non-empty set and the mappings $S,T,f: X \to X$ have a unique point of coincidence v *in* X. If (S,f) and (T,f) are weakly compatible, then S,T and f have a unique common fixed point.

Theorem 2.2 (Azam, Arshad and Beg [2]): Let (X,d) be a cone metric space and the mappings $S,T,f: X \to X$ satisfy: $d(Sx,Ty) \leq \lambda\, d(fx,fy)$ for all $x,y \in X$ where $0 \leq \lambda < 1$. If $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, then S,T and f have a unique point of coincidence. Moreover if (S,f) and (T,f) are weakly compatible, then S,T and f have a unique common fixed point.

In this theorem, if we take $x = y$ then we will get $S = T$. Hence this theorem may be taken as a common fixed point theorem for two maps S and f

Lemma 2.3 (K.P.R.Sastry, Ch.Srinivasa Rao, K.Sujatha and G.Praveena [10]):
Let (X,d) be a complete cone metric space with normal cone P with normal constant K. Suppose $\lambda \in (0,1)$ and $\{x_n\}$ is a sequence in X such that $d(x_n, x_{n+1}) \leq \lambda\, d(x_{n-1}, x_n)$ for n = 1, 2, 3,...

Then $\{x_n\}$ is a Cauchy sequence in X.

Theorem 2.4: Let (X, d) be a cone metric space and the mappings $S, f: X \to X$ satisfy: $d(Sx, Sy) \leq \lambda\, d(fx, fy)$ for all $x, y \in X$ where $0 \leq \lambda < 1$. If $S(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of *X*, then S and f have a unique point of coincidence.

Moreover if $(S, f\ )$ is weakly compatible, then S and *f* have a unique common fixed point.

Proof: Let $x_0 \in X$. Choose a point x_1 in *X* such that $fx_1 = Sx_0$.(since $S(X) \subseteq f(X)$)

Similarly, choose a point x_2 in *X* such that $fx_2 = Sx_1$, $fx_3 = Sx_2$ ….. Continuing this process and having chosen x_n in *X*, we obtain $x_{n+1} \in X$ such that $fx_{n+1} = Sx_n$ and $fx_{n+2} = Sx_{n+1}$.

Then $d\,(fx_{n+1}, fx_{n+2}) = d\,(Sx_n, Sx_{n+1}) \leq \lambda\, d\,(fx_n, fx_{n+1})$

Now by induction, we obtain

$$d\,(fx_{n+1}, fx_{n+2}) \leq \lambda^{n+1} d(fx_0, fx_1)$$

Let $y_n = fx_n$, n = 0, 1, 2,...

Now for all n, we have

$$d(y_{n+1}, y_{n+2}) \leq \lambda\, d(y_n, y_{n+1})$$

Hence by lemma 2.3, $\{y_n\}$ is a Cauchy sequence. since *f (X)* is complete, there exist *u, v* ∈ *X* such that $y_n \to v = fu$. Hence, for n ≥ *N*

$$\begin{aligned}
d(fu, Su) &\leq d(fu, y_{n+1}) + d(y_{n+1}, Su) \\
&= d(v, y_{n+1}) + d(Sx_n, Su) \\
&\leq d(v, y_{n+1}) + \lambda\, d(fx_n, fu) \\
&= d(v, y_{n+1}) + \lambda\, d(y_n, fu) \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

$$\therefore\ d(fu, Su) \leq 0$$

$$\Rightarrow\ d(fu, Su) = 0$$

$$\Rightarrow fu = Su$$

⇒ *v* is a point of coincidence of *S* and *f*.

$$\therefore\ v = fu = Su.$$

Now we show that *f* and *S* have unique point of coincidence. For this, assume that there exists another point v^* in X such that $v^* = fu^* = Su^*$ for some u^* in *X*. Now,

$$d(v, v^*) = d(su, su^*)$$

$\leq \lambda\, d(fu, fu^*)$

$\leq \lambda\, d(v, v^*)$

This implies that $v = v^*$.

If $(S, f\ )$ is weakly compatible, by Lemma 2.1, S and f have a unique common fixed point.

Theorem 2.5: Let $(X, d\ )$ be a cone metric space and the mappings $S, T, f: X \to X$ satisfy:

$$d(Sx, Ty) \leq a\, d(fx, fy) + b\big(d(fx, Sx) + d(fy, Ty)\big) + c\big(d(fy, Sx) + d(fx, Ty)\big) \ldots \qquad (2.5.1)$$

for all $x, y \in X$ where $0\ \leq \lambda < 1$. If $S(X)\ \cup\ T(X) \subseteq\ f(X)$ and $f(X)$ is a complete subspace of X, then S, T and f have a unique point of coincidence. Moreover if $(\ S, f\ )$ and $(\ T, f\ )$ are weakly compatible, then S, T and *f* have a unique common fixed point

Proof: Let $x_0 \in\ X$. Define a sequence of points in X, as follows

$$fx_{2k+1} =\ Sx_{2k},\ fx_{2k+2} =\ Tx_{2k+1},\ \text{k} = 0, 1, 2\ldots\ldots$$

Then, d (f x_{2k+1}, f x_{2k+2}) = d (Sx_{2k}, Tx_{2K+1})

$\leq$ a $d\ (fx_{2k}, fx_{2k+1}) + b\big(d\ (fx_{2k}, Sx_{2k}) + d\ (\ f\ x_{2k+1}, Tx_{2k+1})\big)$

+ c($d\ (fx_{2k+1}, Sx_{2k}) + d\ (\ f\ x_{2k}, Tx_{2k+1})$)

= a $d\ (fx_{2k}, fx_{2k+1}) + b\big(d\ (fx_{2k}, fx_{2k+1}) + d\ (\ f\ x_{2k+1}, fx_{2k+2})\big)$

+ c($d\ (fx_{2k+1}, fx_{2k+1}) + d\ (\ fx_{2k}, fx_{2k+2})$)

= (a + b) $d\ (fx_{2k}, fx_{2k+1}) + b\big(d\ (\ f\ x_{2k+1}, fx_{2k+2})\big)$

+ c ($d\ (\ fx_{2k}, fx_{2k+2})$)

$\leq$ (a + b) $d\ (fx_{2k}, fx_{2k+1}) + b\big(d\ (\ f\ x_{2k+1}, fx_{2k+2})\big)$

+ c($d\ (fx_{2k}, fx_{2k+1}) + d\ (\ fx_{2k+1}, fx_{2k+2})$)

= (a + b + c)$d\ (fx_{2k}, fx_{2k+1}) + (b + c)\big(d\ (\ f\ x_{2k+1}, fx_{2k+2})\big)$

$\Rightarrow d\ (fx_{2k+1}, fx_{2k+2})\ \leq \frac{(\ a+b+c\ )}{(1-(\text{b}+\text{c}))}\ d\ (\ f\ x_{2k}, fx_{2k+1})$

$\Rightarrow d\ (fx_{2k+1}, fx_{2k+2})\ \leq \lambda\ d\ (\ f\ x_{2k}, fx_{2k+1})\ \ldots.$ (2.5.2)

Where $\lambda = \frac{(\ a+b+c\ )}{(1-(\text{b}+\text{c}))} < 1$

Now d (f x_{2k+2}, f x_{2k+3}) = d (Sx_{2k+1}, Tx_{2K+2})

$\leq$ a $d\ (fx_{2k+1}, fx_{2k+2}) + b\big(d\ (fx_{2k+1}, fx_{2k+2}) + d\ (\ f\ x_{2k+2}, Tx_{2k+2})\big)$

$+ \text{c} \,(d\,(fx_{2k+2}, Sx_{2k+1}) + d\,(f\,x_{2k+1}, Tx_{2k+2}))$
$= (\text{a} + \text{b})d\,(fx_{2k+1}, fx_{2k+2}) + b\big(d\,(fx_{2k+2}, fx_{2k+3})\big)$
$+ \text{c}(d\,(fx_{2k+2}, fx_{2k+2}) + d\,(fx_{2k+1}, fx_{2k+3}))$
$\leq (\text{a} + \text{b})d\,(fx_{2k+1}, fx_{2k+2}) + b\big(d\,(fx_{2k+2}, fx_{2k+3})\big)$
$+ \text{c}(d\,(fx_{2k+1}, fx_{2k+2}) + d\,(fx_{2k+2}, fx_{2k+3})$
$\Rightarrow d\,(fx_{2k+2}, fx_{2k+3}) \leq \frac{(a+b+c)}{(1-(\text{b}+\text{c}))}\, d\,(f\,x_{2k+1}, fx_{2k+2})$
$\Rightarrow d\,(fx_{2k+2}, fx_{2k+3}) \leq \lambda\, d\,(f\,x_{2k+1}, fx_{2k+2})$ …. (2.5.3)

Where $\lambda = \frac{(a+b+c)}{(1-(\text{b}+\text{c}))} < 1$

From (2.5.2) and (2.5.3) follows that
$d\,(fx_{n+1}, fx_n) \leq \lambda\, d\,(f\,x_n, fx_{n-1})$, n = 1, 2…… …. (2.5.4)

Let $y_n = fx_n$, n = 0, 1, 2……,
Now for all n = 0, 1, 2……, we have, from (2.5.4)

$$d\,(y_{n+1}, y_{n+2}) \leq \lambda\, d\,(y_n, y_{n+1})$$

∴ By lemma 2.3, $\{y_n\}$ is a Cauchy sequence.

Since $f(X)$ is complete, there exist $u, v \in X$ such that

$y_n \to v = fu$. Hence,
$d(fu, Su) \leq d(fu, y_{2n+2}) + d(y_{2n+2}, Su)$
$= d(v, y_{2n+2}) + d(Tx_{2n+1}, Su)$
$= d(v, y_{2n+2}) + d(Su, Tx_{2n+1})$
$\leq d(v, y_{2n+2}) + \text{a}\, d\,(fu, fx_{2n+1}) + b\big(d\,(fu, Su) + d\,(fx_{2n+1}, Tx_{2n+1})\big)$
$+ \text{c}\,(d\,(fx_{2n+1}, Su) + d\,(f\,u,, Tx_{2n+1}))$
$= d(v, y_{2n+2}) + \text{a}\, d\,(v, y_{2n+1}) + b\big(d\,(fu, Su) + d\,(y_{2n+1}, y_{2n+2})\big)$
$+ \text{c}\,(d\,(y_{2n+1}, Su) + d\,(v, y_{2n+2}))$
(1-b) $d(fu, Su) \leq d(v, y_{2n+2}) + \text{a}\, d\,(v, y_{2n+1}) + b\big(d\,(y_{2n+1}, y_{2n+2})\big)$
$+ \text{c}\,(d\,(y_{2n+1}, Su) + d\,(v, y_{2n+2}))$
$\to d(v, v) + \text{a}\, d\,(v, v) + b\big(d\,(v, v)\big)$
$+ \text{c}\,(d\,(v, Su) + d\,(v, v))$ as $n \to \infty$
$= \text{c}\, d\,(v, Su) = \text{c}\, d\,(fu, Su)$
$\therefore\ (1 - \text{b} - \text{c})\, d(fu, Su) \leq 0$

$\Rightarrow\ d(fu, Su) = 0$ (since 1-b-c is +ve)

$\Rightarrow fu = Su$

Now $d(Su, Tu) =$ a $d\,(fu, Tu) + b\big(d\,(fu, Su) + d\,(fu, Tu)\big)$

$+$ c $(d\,(fu, Su) + d\,(f\,u,, Tu))$

$= (\,a + b + c\,)\,d\,(fu, Tu) +$ (b + c) $d\,(fu, Su)$

$= (\,a + b + c\,)\,d\,(Su, Tu)\,(\,fu = Su\ \&\ d\,(\,fu, fu\,) = 0\,)$

$\Rightarrow (1 - (\,a + b + c\,))\,d\,(Su, Tu) \leq 0$

$\Rightarrow\ d(Su, Tu) = 0$

$\Rightarrow Su = Tu$

$\therefore\ Tu = fu = Su.$

$\therefore$ *u* is a coincidence point of *S, T* and *f*.

Let *w* be any coincidence point of *S, T* and *f*.

$\therefore\ Tw = fw = Sw.$

$d(Su, Tw) \leq$ a $d\,(fu, fw) + b\big(d\,(fu, Sw) + d\,(fu, Tw)\big)$

$+$ c $(d\,(fu, Sw) + d\,(f\,u,, Tw))$

$=$ a $d\,(Su, Tw) + b\big(d\,(Su, Tw) + d\,(Su, Tw)\big)$

$+$ c $(d\,(Su, Tw) + d\,(Su,, Tw))$

$= (\,a + 2b + 2c\,)\,d\,(Su, Tw)$

$\Rightarrow (1 - (\,a + 2b + 2c\,))\,d\,(Su, Tw) \leq 0$

$\Rightarrow\ d(Su, Tw) \leq 0$ (since $(\,a + 2b + 2c\,) < 1$)

$\Rightarrow Su = Tw$

$\therefore$ Coincidence point of *S, T* and *f* is unique.

If further $(S, f\,)$, $(T, f\,)$ are weakly compatible, then by Lemma 2.1, S, T and f have a unique common fixed point.

Now Theorem 2.2, Theorem 2.3 and Theorem 2.4 of (Akaber Azam, Muhammad Arshad and Ismat Beg [2]) became corollaries of our main result (Theorem 2.5)

Corollary 2.6 (Theorem 2.2, [2]): Let $(X, d\,)$ be a cone metric space and the mappings $S, T, f\colon X \to X$ satisfy: $d(Sx, Ty) \leq \lambda\, d(fx, fy)$ for all $x, y \in X$ where $0 \leq \lambda < 1$.

If $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of *X*, then S, T and f have a unique point of coincidence. Moreover if $(\,S, f\,)$ and $(\,T, f\,)$ are weakly compatible, then S, T and *f* have a unique common fixed point.

Proof: Take a = λ, b = 0, c = 0 in Theorem 2.5 then we will get the result.

Corollary 2.7 (Theorem 2.3, [2]): Let (X, d) be a cone metric space and the mappings $S, T, f: X \to X$ satisfy: $d(Sx, Ty) \leq \lambda(d(fx, Sx) + d(fy, Ty))$ for all $x, y \in X$ where $0 \leq \lambda < \frac{1}{2} < 1$. If $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of *X*, then S, T and f have a unique point of coincidence. Moreover if (S, f) and (T, f) are weakly compatible, then S, T and *f* have a unique common fixed point.

Proof: Take a = 0, b = λ, c = 0 in Theorem 2.5 then we will get the result.

Corollary 2.8 (Theorem 2.4, [2]): Let (X, d) be a cone metric space and the mappings $S, T, f: X \to X$ satisfy: $d(Sx, Ty) \leq \lambda (d(fy, Sx) + d(fx, Ty))$ for all $x, y \in X$ where $0 \leq \lambda < 1$. If $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of *X*, then S, T and f have a unique point of coincidence. Moreover if (S, f) and (T, f) are weakly compatible, then S, T and *f* have a unique common fixed point.

Proof: Take a = 0, b = 0, c = λ in Theorem 2.5 then we will get the result.

Example 2.9: If E = R × R, with cone P = { (x, y) / $x \geq 0, y \geq 0$ }, X = R^+ (The set of non- negative real numbers). Define d: X → P by d (x, y) = (/ x-y /,$\frac{1}{2}$/ x-y /) ∀ (x, y) ∈ X

Then (X, d) be a cone metric space. Define the mappings $S, T, f: X \to X$ satisfy:

$$d(Sx, Ty) \leq \frac{1}{4}\, d(fx, fy) + \frac{1.25}{7}\big(d(fx, Sx) + d(fy, Ty)\big) + \frac{1.25}{7}\big(d(fy, Sx) + d(fx, Ty)\big)$$

for all $x, y \in X$. If $S(X) \cup T(X) \subseteq f(X)$ and $f(X)$ is a complete subspace of *X* such that $S\,x$ = T $x = x$ and $f\,x = 2\,x$ for all $x \in R^+$.

Then S, T and f have a unique point of coincidence. Moreover if (S, f) and (T, f) are weakly compatible, then S, T and *f* have a unique common fixed point

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