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A Third Order Boundary Value Problem with Nonlinear Terms-Existence and Uniqueness

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Abstract

In this paper, we are concerned with the following third – order boundary value problem:

$u'''(t) + f\big(t, u(t), u'(t), u''(t)\big) = 0, t \in [0,1],$
$u(0) = 0, u'(0) = 0, u'(1) = \phi u'(\psi),$

Where $f: [0,1] \times R^3 \to R$ is continuous, $\phi > 0, 0 < \psi < 1$ such that $\phi\,\psi < 1$. By using two pairs of lower and upper solutions method of Henderson and Thompson and Leray Schauder degree theory, the existence result of at least three solutions for the problem is given.

Mathematics Subject Classification: 34B05, 34B99

Keywords: Third–order boundary value problem, Leray-Schauder degree theory, lower and upper solutions.

Introduction

In this paper, we deals with the multiplicity of solutions for the following third order boundary value problem

$$u'''(t) + f\big(t, u(t), u'(t), u''(t)\big) = 0, t \in [0,1] \tag{1.1}$$

$$u(0) = 0, u'(0) = 0, u'(1) = \phi u'(\psi) \tag{1.2}$$

Throughout this paper, we suppose that $\phi > 0, 0 < \psi < 1$ such that $\phi\,\psi < 1$ and $f: [0,1] \times R^3 \to R$ is continuous. Here, we apply two pairs of lower and upper solutions method of Henderson and Thompson [4] to study the boundary value problem(1.1), (1.2). Under the condition that f(t,u,v,w) satisfies a Nagumo condition, we obtain the existence of three solutions by use of Leray-Schauder degree theory.

Notations and Definitions

We shall use the classical spaces $C[0,1], C^2[0,1]$ and $L^1[0,1]$. For x $\in C^2[0,1]$, we use the norm $\|x\|_\infty = max\{|x(t)|: t \in [0,1]\}$ and $\|x\| = max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty\}$. We will use the Sobolev space $W^{3,1}(0,1)$ which defined by

$$W^{3,1}(0,1) = \{x: [0,1] \to R / \; x, x', x'' \; are\ absolutely\ continuous\ on\ [0,1] with\ x''' \in L^1[0,1]\}.$$

Definition 2.1

A function $\rho(t) \in W^{3,1}(0,1)$ is called a lower solution for the problem (1.1), (1.2) if

$$\rho'''(t) + f(t, \rho(t), \rho'(t), \rho''(t)) \geq 0, 0 < t < 1 \quad (2.1)$$

$$\rho(0) \leq 0, \rho'(0) \leq 0, \rho'(1) \leq \phi\rho'(\psi) \quad (2.2)$$

Similarly, a function $\sigma(t) \in W^{3,1}(0,1)$ is called an upper solution for the problem (1.1), (1.2) if

$$\sigma'''(t) + f(t, \sigma(t), \sigma'(t), \sigma''(t)) \leq 0, 0 < t < 1 \quad (2.3)$$

$$\sigma(0) \geq 0, \sigma'(0) \geq 0, \sigma'(1) \geq \phi\sigma'(\psi) \quad (2.4)$$

Definition 2.2 let ρ be a lower solution and σ upper solution for the problem (1.1), (1.2) satisfying $\rho \leq \sigma$ and $\rho' \leq \sigma'$ on [0,1]. We say that f satisfies a Nagumo condition with respect to $\rho\ and\ \sigma$, if there exists a function $\Psi \in C([0,\infty); (0,\infty))$ such that

$$|\mathrm{f(t,u,v,w)}| \leq \Psi(|w|) \quad (2.5)$$

For all $(\mathrm{t,u,v,w}) \in [0,1] \times [\rho(t), \sigma(t)] \times [\rho'(t), \sigma'(t)] \times R$ and

$$\int_0^\infty \frac{s}{\Psi(s)} ds = \infty \quad (2.6)$$

$It\ is\ clear\ that$, we can give the Green's function G(t,s) of the problem (1.1), (1.2) since the boundary conditions satisfies 1 - $\phi\ \psi$ > 0.

Existence of Triple Solutions

Theorem 3.1 Assume that

There exists two lower solutions ρ_1, ρ_2 and upper solutions σ_1, σ_2 of the problem (1.1), (1.2) satisfying $\rho_1{}' \leq \rho_2{}' \leq \sigma_2{}', \rho_1{}' \leq \sigma_1{}' \leq \sigma_2{}', \rho_2{}' \nleq \sigma_1{}'$ on [0,1]

Let f(t,u,v,w): $[0,1] \times R^3 \to R$ be a continuous function and non decreasing with respect to u, for $(t, u, v, w) \in [0,1] \times [\rho_1(t), \sigma_2(t)] \times R^2$;

F satisfies Nagumo condition with respect to $\rho_1\ and\ \sigma_2$, the problem (1.1), (1.2) has at least three solutions $u_1, u_2\ and\ u_3$ satisfying

$\rho_1 \le u_1 \le \sigma_1, \rho_1{}' \le u_1{}' \le \sigma_1{}'$ on [0,1] and $\rho_2 \le u_2 \le \sigma_2, \rho_2{}' \le u_2{}' \le \sigma_2{}'$ on [0,1], $u_3 \nleq \sigma_1, u_3{}' \nleq \sigma_1{}'$ and $u_3 \ngeq \rho_2, u_3{}' \ngeq \rho_2{}'$ on [0,1].

Proof: From assumption (iii), we can choose C > 0, such that

$$\int_\lambda^C \frac{s}{\Psi(s)} ds = \lambda \tag{3.1}$$

$\lambda = \max_{t\in[0,1]} \sigma_2{}'(t) - \min_{t\in[0,1]} \rho_1{}'(t)$. Let $L = \max\left\{\left\|\rho_1{}''\right\|_\infty, \left\|\sigma_2{}''\right\|_\infty, \mathrm{C}, 2\lambda\right\}$. We define three auxiliary functions f_1, f_2 and $F: [0,1] \times R^3 \to R$ as

$$f_1(t,u,v,w) = \begin{cases} f(t,\sigma_2,v,w), u > \sigma_2(t), t \in [0,1] \\ f(t,u,v,w),\ \rho_1(t) \le u \le \sigma_2(t), t \in [0,1] \\ f(\mathrm{t},\rho_1,\mathrm{v},\mathrm{w}), \mathrm{u} < \rho_1(t), t \in [0,1] \end{cases} \tag{3.2}$$

$$f_2(t,u,v,w) = \begin{cases} f_1(t,u,\sigma_2{}',w), v > \sigma_2{}'(t), t \in [0,1] \\ f_1(t,u,v,w), \rho_1{}'(t) \le v \le \sigma_2{}'(t), t \in [0,1] \\ f_1(\mathrm{t},u,\rho_1{}',\mathrm{w}), \mathrm{v} < \rho_1{}'(t), t \in [0,1] \end{cases} \tag{3.3}$$

$$F(t,u,v,w) = \begin{cases} f_2(t,u,v,L), w > L, t \in [0,1] \\ f_2\ \ (t,u,v,w), |w| \le L, t \in [0,1] \\ f_2(\mathrm{t},u,v,-\mathrm{L}), \mathrm{w} < -L, t \in [0,1] \end{cases} \tag{3.4}$$

Thus F is continuous function on $[0,1] \times R^3$, satisfying

$$|F(t,u,v,w)| \le M, \text{ for (t,u,v,w)} \in [0,1] \times R^3 \tag{3.5}$$

Where constant M also satisfies M > $max\{\|\rho_1\|_\infty, \|\sigma_2\|_\infty\}$.
Consider the modified problem

$$u'''(t) + F\left(t,u(t),u'(t),u''(t)\right) = 0, t \in [0,1] \tag{3.6}$$

With the boundary condition (1.2). it suffices to show that problem (3.6) with the boundary condition (1.2) has at least three solutions u_1, u_2 *and* u_3 satisfying

$$\rho_1(t) \le u_i \le \sigma_2(t), \rho_1{}'(t) \le u_i{}'(t) \le \sigma_2{}'(t),\ u_i{}''(t) \le L, t \in [0,1] \tag{3.7}$$

Since F = f in the region, we divide the proof in to cases.

Case I: First we show that $\rho_1{}' \le u' \le \sigma_2{}'$ on [0,1]. We only need to show $u' \le \sigma_2{}'$ on [0,1]. Similarly, we can prove $\rho_1{}' \le u'$ on [0,1]. If $u' \le \sigma_2{}'$ on [0,1] is not true, then there exists $t \in [0,1]$ *with* $u' > \sigma_2{}'$. Now sot $\omega(t) = u'(t) - \sigma_2{}'(t)$, then $\omega(t_0) = \max\{u'(t) - \sigma_2{}'(t): t \in [0,1]\} > 0$ for some $t_0 \in [0,1]$.

Sub Case 1. If $t_0 = 0$, then $u'(0) \le \sigma_2{}'(0)$, from (2.4), we have the contradiction $\sigma_2{}'(0) \ge 0 = u'(0)$.

Sub Case 2. If $t_0 \in [0,1]$, we have $\omega(t_0) > 0$, $\omega'(t_0) = 0, \omega''(t_0) \le 0$. On the other hand, $\omega''(t_0) = -F\left(t_0, u(t_0), u'(t_0), u''(t_0)\right) - \sigma_2'''(t_0)$ $= -f_2\left(t_0, u(t_0), u'(t_0), \sigma_2''(t_0)\right) - \sigma_2'''(t_0)$ $= -f_1\left(t_0, u(t_0), \sigma_2'(t_0), \sigma_2''(t_0)\right) - \sigma_2'''(t_0)$.

Step 1. If $u(t_0) > \sigma_2(t_0)$, from the above inequality, one has

$$\omega''(t_0) = -f_1\left(t_0, u(t_0), \sigma_2'(t_0), \sigma_2''(t_0)\right) - \sigma_2'''(t_0)$$
$$= -f_1\left(t_0, \sigma_2(t_0), \sigma_2'(t_0), \sigma_2''(t_0)\right) - \sigma_2'''(t_0) > 0.$$

Step 2. If $u(t_0) \leq \sigma_2(t_0)$, from the above inequality and (Y_2),

$$\omega''(t_0) = -f_1\left(t_0, u(t_0), \sigma_2'(t_0), \sigma_2''(t_0)\right) - \sigma_2'''(t_0)$$
$$= -f_2\left(t_0, \sigma_2(t_0), \sigma_2'(t_0), \sigma_2''(t_0)\right) - \sigma_2'''(t_0) > 0.$$ This is a contradiction.

Sub Case 3. If $t_0 = 1$, then $\omega(1) > 0$. (3.8)

From (2.4), we have $\omega(0) \leq 0$, thus there exists $\xi \in [0,1]$ such that

$\omega(\xi) = 0$ and $\omega(t) > 0$ for all $t \in [\xi, 1]$. (3.9)

If $\xi \in (\psi, 1)$, then there exists $t_1 \in (0, \xi)$ such that $\omega(t_1) = \max\{\omega(t): t \in (0, \xi)\}$.

From (1.2), (2.4) and (3.8), we have $\omega(t_1) \geq \omega(\psi) = u'(\psi) \leq \sigma_2'(\psi)$

$$\geq \frac{1}{\phi}[u'(1) \leq \sigma_2'(1)] = \frac{1}{\phi}\omega(1) > 0.$$

Moreover, $\omega'(t_1) = 0$ *and* $\omega''(t_1) \leq 0$. similar to the Sub case 2, we have contradiction.

If $\xi \in (0, \psi)$, then for all $t \in [\xi, 1]$, we have that $\omega(t) \geq 0$. We consider the following two steps: Step(i) $\omega'(t) \geq 0, t \in [\xi, 1]$; and Step(ii), there exists some $t_2 \in [\xi, 1]$ such that $\omega(t_2) > 0$, $\omega'(t_2) = 0, \omega''(t_2) \leq 0$. For step (i), similar to Sub Case(2), we have $\omega''(t) > 0$ or $\omega(t) > 0$, $\omega'(t_2) = 0, \omega''(t) > 0$ for all $t \in [\xi, 1]$, which implies that the graph of ω is concave upward on $[\xi, 1]$, *and also* $\frac{\omega(\psi)}{\psi} < \frac{\omega(1)}{1}$.

On the other hand, we have $\omega(1) = u'(1) - \sigma_2'(1) \leq \phi[u'(\psi) - \sigma_2'(\psi)] = \phi\omega(\psi)$,

From $0 < \phi < 1/\psi$, we obtain $\frac{\omega(\psi)}{\psi} \geq \frac{\omega(1)}{1}$. This is contradiction.

For step (i), similar to the argument of Sub Case (2), we have contradiction. Thus $u' \leq \sigma_2'$ on [0,1], then $\rho_1' \leq u' \leq \sigma_2'$ on [0,1].Since $\rho(0) \leq 0, \sigma(0) \geq 0$, by integrating the above inequalities on [0,t], we obtain $\rho_1 \leq u \leq \sigma_2$ on [0,1].

We show that $|u''| \leq L$ on [0,1]. If the assertion is not true, without loss of generality, we suppose that there exist $t \in [0,1]$, satisfying $u''(t) > L$ attains its positive maximum over [0,1]. From mean value theorem and $\rho_1{}' \le u' \le \sigma_2{}'$ on [0,1], there exists $\theta \in [0,1]$, such that $u''(\theta) = u'(1) - u'(0) \le \eta < L$. Since $u''(t) \in C[0,1]$, then there exists interval $[t_{4,} t_5] \subseteq [0,1]$ such that

$$u''(t_{4,}) = \eta,\ u''(t_5) = L, \eta < u''(t) < L, t \in (t_{4,} t_5). \tag{3.10}$$

Thus from (2.5), we obtain $|u'''(t)| = |F(t,u,u',u'')| = |f(t,u,u',u'')| \le \Psi(u'')$, $t \in (t_{4,} t_5)$, then $\left|\int_{t_{4,}}^{t_5} \frac{u''(t)u'''(t)}{\Psi(u''(t))} dt\right| \le \left|\int_{t_4}^{t_5} u''(t)\, dt\right| \le \eta.$ (3.11)

From (3.1), (3.10), we have $\left|\int_{t_{4,}}^{t_5} \frac{u''(t)u'''(t)}{\Psi(u''(t))} dt\right| = \left|\int_{\eta}^{L} u''(t)\, dt\right| > \eta.$ (3.12)

Then (3.11) contradicts (3.12). Thus $|u''| \le L$ on [0,1]. Thus x is the required solution.

Case II. We show that the problem (3.6) and (2) has at three solutions $u_{1,}\ u_{2,}\ and\ u_3$.

Let $\Omega = \{u \in C^2[0,1]: \|u\| < PM + L\}$, Where $P > max\left\{\max_{t\in[0,1]} \int_0^1 |G(t,s)|\, ds, 1\right\}$, G(t,s) is a Green's function of the problem (1.1), (1.2). Define, $S: C[0,1] \to C^2[0,1]$ by $S\phi(t) = \int_0^1 G(t,s)\, \phi(s) ds$, For all $\phi \in C[0,1]$ and $t \in [0,1]$. It is clear that S is completely continuous. Define $H: C[0,1] \to C^2[0,1]$ as $H\phi(t)$ = F(t, $\phi(t), \phi'(t), \phi''(t)$).

Then, $u \in C^2[0,1]$ is a solution of the problem (3.6) and (2) if and only if (I-SH) (u) = 0. For $u \in \overline{\Omega}$, we have $SH(x) = \int_0^1 \mathrm{G(t,s)\ F}(s, u(s), u'(s), u''(s))\mathrm{ds}.$

$\le M \int_0^1 G(t,s) ds$ < PM<PM+L

Clearly, $SH(\overline{\Omega}) \subset \Omega$ and SH is completely continuous. Then we have

$$\deg(I - SH, \Omega, 0) = deg(I, \Omega, 0) = 1. \tag{3.13}$$

Let, $\Omega_{\rho 2} = \{u \in \Omega: u' > \rho_2{}' \text{ on } (0,1)\}$, $\Omega^{\sigma 1} = \{u \in \Omega: u' > \sigma_1{}' \text{ on } (0,1)\}$.

Since $\rho_2{}' \nleq \sigma_1{}', \rho_2{}' \ge \sigma_1{}' > -L$ and $\sigma_1{}' \le \sigma_2{}' < L$, it follows that

$\Omega_{\rho 2} \ne \emptyset \ne \Omega^{\sigma 1}, \overline{\Omega_{\rho 2}} \cap \overline{\Omega}^{\sigma_1} = \emptyset, \Omega \big/ \overline{\Omega_{\rho 2} \cup \Omega^{\sigma 1}} \ne 0.$

But no solution on $\partial\Omega_{\rho 2} \cup \partial\Omega^{\sigma 1}$. Thus

$$\deg(I - SH, \Omega, 0) = \deg\left(I - SH, \Omega \big/ \overline{\Omega_{\rho 2} \cup \Omega^{\sigma 1}}, 0\right) + \deg(I - SH, \Omega^{\sigma 1}, 0)$$

$+\deg(I - SH, \partial\Omega_{\rho 2}, 0)$.

If we prove that $\deg(I - SH, \Omega^{\sigma 1}, 0) = \deg(I - SH, \Omega_{\rho 2}, 0) = 1$, Then $\deg\left(I - SH, \,\Omega\big/_{\Omega_{\rho 2} \,\cup\, \Omega^{\sigma 1}}, 0\right)$ = -1 and hence there are solutions in $\Omega_{\rho 2}, \Omega^{\sigma 1}$ and $\Omega\big/_{\Omega_{\rho 2} \,\cup\, \Omega^{\sigma 1}}$ respectively. We show that $\deg(I - SH, \Omega_{\rho 2}, 0) = 1$. Therefore the proof that $\deg(I - SH, \Omega^{\sigma 1}, 0) = 1$ is the same and hence omitted. Similar to the conditions of f_1, we

define $f_1{}^*(t,u,v,w) = \begin{Bmatrix} f(t,\sigma_2,v,w), u > \sigma_2(t), t \in [0,1]; \\ f(t,u,v,w), \rho_2(t) \le u \le \sigma_2(t), t \in [0,1]; \\ f(t,\rho_2,v,w), u < \rho_2(t), t \in [0,1]. \end{Bmatrix}$

$$f_2{}^*(t,u,v,w) = \begin{Bmatrix} f_1{}^*(t,u,\sigma_2{}'(t),w), v > \sigma_2{}'(t), t \in [0,1]; \\ f_1{}^*(t,u,v,w), \rho_2{}'(t) \le v \le \sigma_2{}'(t), t \in [0,1]; \\ f_1{}^*(t,u,\rho_2{}'(t),w), v < \rho_2{}'(t), t \in [0,1]. \end{Bmatrix}$$

Now from I - SH $/\overline{\Omega_{\rho 2}}$, we define its extension I - SH^*: $\overline{\Omega} \to C^2[0,1]$ as follows.

$$F^*(t,u,v,w) = \begin{Bmatrix} f_2{}^*(t,u,v,L), w > L, t \in [0,1]; \\ f_2{}^*(t,u,v,w), |w| \le L, t \in [0,1]; \\ f_2{}^*(t,u,v,-L), w < -L, t \in [0,1]. \end{Bmatrix} \tag{3.14}$$

Thus $F^*$is a continuous function on $[0,1] \times R^3$ and satisfies

$$|F^*(t,u,v,w)| \le M, \tag{3.15}$$

For all $(t,u,v,w) \in [0,1] \times R^3$, where M is given in (3.5).

Define $\boldsymbol{H}^*$: $C^2[0,1] \to C[0,1]$ as follows $H^*\phi(t) = F^*(t, \phi(t), \phi'(t), \phi''(t))$.

Then, $u \in C^2[0,1]$ is a solution of (I - S$H^*)(u) = 0$ if and only if u is a solution of

$$u'''(t) + F^*(t,u,u',u'') = 0, t \in [0,1] \tag{3.16}$$

Similar to the above argument, it follows that u is a solution of (3.16) with (2) only if u $\in \Omega_{\rho 2}$. Thus $\deg(I - SH^*, \Omega\backslash\overline{\Omega_{\rho 2}}) = 0$. Similarly, we show that $SH^*(\overline{\Omega}) \subset \Omega$. Then we have, $\deg(I - SH^*, \Omega, 0) = 1$. Thus $\deg(I - SH, \Omega_{\rho 2}, 0) = \deg(I - SH^*, \Omega_{\rho 2}, 0) = \deg(I - SH^*, \Omega\backslash\overline{\Omega_{\rho 2}}, 0) + \deg(I - SH^*, \Omega_{\rho 2}, 0) = \deg(I - SH^*, \Omega, 0) = 1$.

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