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A Relation between Two Summability Methods

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Abstract

In this paper a relation on φ - $|C, 1|_k$ and φ - $|\bar{N}, p_n|_k$ summability methods is established.

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Introduction

Let $A = [a_{nk}]$ be an infinite matrix of real or complex numbers a_{nk} (n, k = 1, 2 , …….) and let $\{\varphi_n\}$ be a sequence of real or complex numbers. Let Σa_n be a given infinite series with the sequence of partial sums $\{s_n\}$. By $A_n(s)$ we denote the A – transform of the sequence $s = \{s_n\}$, that is

$$A_n(s) = \sum_{r=1}^{\infty} a_{nr} s_r . \tag{1.1}$$

The series Σa_n is summable $|A|$, if

$$\sum_{n=1}^{\infty} | A_n(s) - A_{n-1}(s) | < \infty , \tag{1.2}$$

and it is said to be summable φ - $|A|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} | \varphi_n [A_n(s) - A_{n-1}(s)] |^k < \infty . \tag{1.3}$$

Let u_n and t_n denote nth (C, 1) mean of the sequence $\{s_n\}$ and $\{na_n\}$ respectively. Then the series Σa_n is said to be summable φ - $|C, 1|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \left|\varphi_n (u_n - u_{n-1})\right|^k < \infty . \tag{1.4}$$

Since $t_n = n\,(u_n - u_{n-1})$ so the condition (1.1) can restated as

$$\sum_{n=1}^{\infty} \frac{\left|\varphi_n t_n\right|^k}{n^k} < \infty \tag{1.5}$$

(If $\varphi_n = n^{1-\frac{1}{k}}$ then it is same as $|C, 1|_k$ summability method.).

Let $\{ p_n \}$ be a sequence of positive real constants such that

$$P_n = \sum_{r=0}^{n} p_r \quad \to \infty \text{ as } n \to \infty \ (P_{-1} = p_{-1} = 0). \tag{1.6}$$

A series Σa_n is said to be summable φ - $|\bar{N}, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \left|\varphi_n (T_n - T_{n-1})\right|^k < \infty , \tag{1.7}$$

where

$$T_n = \frac{1}{P_n} \sum_{r=1}^{n} p_r a_r$$

(It may be noted that if we take $\varphi_n = n^{1-\frac{1}{k}}$ then it is same as $|\bar{N}, p_n|_k$ summability method).

Preliminary Result

Bor [2] established a relation between $|C, 1|_k$ and $|\bar{N}, p_n|_k$ summability methods. The theorem is as follows:

Theorem 1: Let $\{ p_n \}$ be a sequence of positive real constants such that as $n \to \infty$

$$n\, p_n = O (P_n), \tag{2.1}$$

$$P_n = O (n\, p_n). \tag{2.2}$$

If Σa_n is said to be summable $|C, 1|_k$ then it is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

Main Result

The main object of this paper is to establish a relation between φ - $|C, 1|_k$ and φ - $|\bar{N}$, $p_n|_k$ summability methods. The theorem is

Theorem : *Let { p_n } be a sequence of non-decreasing positive real constants such that it satisfies the conditions (2.1) and (2.2). For $\in > 0$, if the sequence { $n^{\in-k}\,|\varphi_n|^k$ } is non – increasing and Σa_n is φ -$|C, 1|_k$ summable then it is φ -$|\bar{N}$, $p_n|_k$, $k \geq 1$ summable.*

Proof of the Theorem

Let t_n denote the nth (C,1) mean of the sequence { na_n }. So we have

$$t_n = \frac{1}{n+1}\sum_{r=0}^{n} ra_r \,. \tag{4.1}$$

Since Σa_n is φ – $|C, 1|_k$ summable, by (1.5)

$$\sum_{n=1}^{\infty}\frac{|\varphi_n t_n|^k}{n^k} < \infty \,.$$

Let T_n be the nth ($\bar{N}$, p_n) mean of the sequence { s_n }. So we have

$$T_n = \frac{1}{P_n}\sum_{r=1}^{n} p_r a_r \quad = \frac{1}{P_n}\sum_{r=0}^{n}(P_n - P_{r-1})a_r \quad . \tag{4.2}$$

Then

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{r=1}^{n} P_{r-1} a_r$$

$$= \frac{p_n}{P_n P_{n-1}}\sum_{r=1}^{n} r^{-1} P_{r-1} r a_r \quad . \tag{4.3}$$

Applying Abel's partial summation formula to (4.3)

$$T_n - T_{n-1} = \frac{-p_n}{P_n P_{n-1}}\sum_{r=1}^{n-1} p_r t_r \quad + \frac{p_n}{P_n P_{n-1}}\sum_{r=1}^{n-1} P_{r-1} r^{-1} t_r \quad +(n+1)\, p_n\, (nP_n)^{-1}\, t_n$$

$= T_{n,1} + T_{n,2} + T_{n,3}$ (say) [3].

To prove this theorem by Minkowski's inequality it is sufficient to show that

$$\sum_{n=1}^{\infty}|\varphi_n \; T_{n,r}|^k < \infty \text{, for r = 1, 2, 3.}$$

Now applying Hölder's inequality with indices k and k′ where $\frac{1}{k}+\frac{1}{k'}=1$ we have

$$\sum_{n=1}^{m+1}\left|\varphi_n \;\; T_{n\;,\;1}\right|^k \;\; \le \sum_{n=2}^{m+1}\left[\frac{\left|\varphi_n\right|^k p_n^k}{P_n^k P_{n-1}}\left[\sum_{r=1}^{n-1} p_r \,|\, t_r \,|^k\right]\right] \;\; X \left[\frac{1}{P_{n-1}}\sum_{r=1}^{n-1} p_r\right]^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1}\left[\frac{\left|\varphi_n\right|^k}{n^k P_{n-1}} \quad \sum_{r=1}^{n-1} p_r \,|\, t_r \,|^k\right] \tag{4.4}$$

$$= O(1) \sum_{r=1}^{m} p_r \,|\, t_r \,|^k \;\; \sum_{n=r+1}^{m+1}\frac{\left|\varphi_n\right|^k}{n^k P_{n-1}}$$

$$= O(1) \sum_{r=1}^{m} p_r \;\; |\, t_r \,|^k \;\; r^{\epsilon-k} \;\; \left|\varphi_r\right|^k \;\; \sum_{n=r+1}^{m+1}\frac{1}{n^{\epsilon} P_{n-1}}$$

$$= O(1) \sum_{r=1}^{m} |\, t_r \,|^k \;\; r^{\epsilon-k} \;\; \left|\varphi_r\right|^k \;\; \sum_{n=r+1}^{m+1}\frac{1}{n^{\epsilon+1}}$$

$$= O(1) \sum_{r=1}^{m}\frac{|\, t_r \varphi_r \,|^k}{r^k}$$

$$= O(1)$$

in view of (1.5).

Next

$$\sum_{n=1}^{m+1}\left|\varphi_n \;\; T_{n,2}\right|^k \;\; \le \sum_{n=2}^{m+1}\left[\frac{\left|\varphi_n\right|^k p_n{}^k}{P_n^k P_{n-1}^k}\left[\sum_{r=1}^{n-1}\frac{p_{r-1}^k}{r^k} \,|\, t_r \,|^k\right] \;\; \right]$$

$$= O(1) \sum_{n=2}^{m+1}\left[\frac{\left|\varphi_n\right|^k}{n^k P_{n-1}^k} \quad \sum_{r=1}^{n-1} p_{r-1}^k \,|\, t_r \,|^k\right]$$

$$= O(1) \sum_{r=1}^{m}\frac{\left|\varphi_r\right|^k}{n^k P_{n-1}}\sum_{r=1}^{n-1} p_{r-1} \,|\, t_r \,|^k \;\; \times \left[\frac{1}{P_{n-1}} \quad \sum_{r=1}^{n-1} p_{r-1} \quad\right]^{k-1}$$

$$= O(1) \sum_{r=1}^{m}\frac{\left|\varphi_r\right|^k}{n^k P_{n-1}}\sum_{r=1}^{n-1} p_{r-1} \,|\, t_r \,|^k$$

$$= O(1)$$

as in case of (4.4).

Finally

$$\sum_{n=1}^{m+1} \left|\varphi_n \quad T_{n,3}\right|^k \quad \le \sum_{n=2}^{m+1} \frac{\left|\varphi_n\right|^k (n+1)^k p_n^k}{n^k P_n^k} \,|\, t_r \,|^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{\left|\varphi_n t_n\right|^k}{n^k}$$

$$= O(1)$$

in view of (1.5).

Thus the theorem is established.

References

[1] Balci, M. Absolute φ – summability factors, Comm. Fac. Sci. Univ. Ankara, Ser. A_1, 29(1980), 63-68

[2] Bor, H. On Two Summability Methods, Math. Proc. Camb. Phil. Soc. 97 (1985), 147 – 149.

[3] N.Kishore, Misra U.K, Sahu, J. K. On $|N, p_n|_k$ -Summability Factors of Infinite Series, The Aligarh Bull. of Math. 16 (1995-96) 61 - 69.