Some New Separation Axioms: A Different Approach

Chandan Chattopadhyay

Department of Mathematics, Narasinha Dutt College
Howrah, West Bengal, India
E-mail: chandanndc@rediffmail.com

Abstract

This paper studies some new separation axioms for topological spaces defined in terms of a new topology and a continuous function. This new idea gives the notion of Star-T$_1$-spaces, Star-T$_2$-spaces, Star-regular spaces, Star-normal-spaces etc. It is found that Star-T$_1$-axiom lies between T$_0$-axiom and T$_1$-axiom. T$_2$-axiom implies Star-T$_2$-axiom but not conversely. Moreover, T$_1$-axiom and Star-T$_2$-axiom are independent of each other. This is justified with an example that a co-finite topology cannot be Star-T$_2$. There are two types of star-regular-spaces, one is A-star-regular and another is B-star-regular. It is found that a regular space is both A-star-regular and is B-star-regular. On the other hand, a finite B-star regular space is regular.

Finally, it is observed that for a compact T$_1$-space, all the three separation axioms viz., A-star-regular, B-star-regular and star-normal are equivalent.

Keywords: separation axioms, star-regular, star-normal, co-finite, compact.

AMS Subject Classification: 54A05.

Introduction and preliminaries

With the introduction of generalized open sets, like semi-open sets[6], locally closed sets[2] pre-open sets[7], $\delta$-sets[3], semi-pre-open sets[1] etc., separation axioms using these generalized open sets have been defined and the corresponding topological spaces are studied. So far there have been two ways of defining new separation axioms in topological spaces. one, in terms of “each singleton satisfying certain conditions” and the other, in terms of “generalized open sets”. Following are definitions of some known separation axioms.

DEFINITION 1.1 [2] A topological space $(X, t)$ is called a T$_D$-space if each
singleton is locally closed.

**Definition 1.2** [4, 5] A topological space \((X, t)\) is called \(R_0\) if for each open set \(O\) in \(t\), \(x \in O\) implies \(\text{cl } x \subseteq O\).

**Definition 1.3** [3] A topological space \((X, t)\) is called \(T_\partial\)-space if each singleton is a \(\partial\)-set in \((X, t)\).

**Definition 1.4** [4,5] A topological space \((X, t)\) is called \(R_1\) if for \(x, y \in X\) such that \(\text{cl } x \neq \text{cl } y\), there exists disjoint open sets \(U\) and \(V\) such that \(\text{cl } x \subseteq U\) and \(\text{cl } y \subseteq V\).

**Definition 1.5** [6] A topological space \((X, t)\) is called semi-To if for \(x, y \in X\) there exists a semi-open set \(U\) such that \(x \in U\) and \(y\) does not belong to \(U\) or \(y \in U\) and \(x\) does not belong to \(U\).

**Definition 1.6** [6] A topological space \((X, t)\) is called semi-T1 if for \(x, y \in X\) there exist semi-open sets \(U\) and \(V\) such that \(x \in U\) and \(y \in V\) and \(x\) does not belong to \(V\) and \(y\) does not belong to \(U\).

**Definition 1.7** [6] A topological space \((X, t)\) is called semi-T2 if for \(x, y \in X\) there exist disjoint semi-open sets \(U\) and \(V\) such that \(x \in U\) and \(y \in V\).

In the present approach, a new topology and a continuous function have been used.

**Star-\(T_1\) spaces**

**Definition 2.1** A topological space \((X, t)\) is said to be star-\(T_1\) if there exists a topology \(s\) on \(X\) and a bijective continuous function \(f: (X, t) \rightarrow (X, s)\) such that for any two distinct points \(a, b\) in \(X\) there are open sets \(G\) and \(H\) in \(s\) satisfying the following condition:

\[
\begin{align*}
\text{a} & \in f^{-1}(G), \quad \text{b} \in H \quad \text{and} \quad \text{b} \not\in f^{-1}(G), \quad \text{a} \not\in H \\
\text{or} \\
\text{b} & \in f^{-1}(G), \quad \text{a} \in H \quad \text{and} \quad \text{a} \not\in f^{-1}(G), \quad \text{b} \not\in H.
\end{align*}
\]

**Note 2.1** If \((X, t)\) is \(T_1\) then it is star-\(T_1\).

For, consider \(s = t\) and \(f\) the identity function.

**Note 2.2** A star-\(T_1\) -space may not be \(T_1\).

**Example 2.1** Let \(X = \{a, b, c\}\), \(t = \{\emptyset, X, \{a, b\}, \{b\}, \{b, c\}\}\), \(s = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}\).

**Define** \(f: (X, t) \rightarrow (X, s)\) by \(f(a) = c\), \(f(b) = a\), \(f(c) = b\). Then \(f\) is bijective and continuous. Now for the points \(a, b\), choose \(G = \{a\}, H = \{a, c\}\).
Then $b \in f^{-1}(G)$, $a \in H$, $a \not\in f^{-1}(G)$, $b \not\in H$. For the points $a$, $c$, choose $G = \{a, b\}$, $H = \{a\}$. Then $c \in f^{-1}(G)$, $a \in H$, $a \not\in f^{-1}(G)$, $c \not\in H$.

For the points $b$, $c$, choose $G = \{a\}$, $H = \{a, c\}$. Then $b \in f^{-1}(G)$, $c \in H$, $c \not\in f^{-1}(G)$, $b \not\in H$.

Thus $(X, t)$ is star-$T_1$ but not $T_1$.

**NOTE 2.3** A star-$T_1$ -space is $T_0$.

For, $f$ being continuous, $f^{-1}(G)$ is open in $t$. Now the result follows.

**NOTE 2.4** A $T_0$ -space may not be star-$T_1$.

**EXAMPLE 2.2** Let $X = \{a, b\}$, $t = \emptyset, X, \{a\}$. Then $(X, t)$ is $T_0$. We claim that $(X, t)$ is not star-$T_1$.

If possible, let $(X, t)$ be star-$T_1$. Then there exists a topology $s$ on $X$ and a bijective continuous function $f: (X, t) \to (X, s)$ such that for any two distinct points $a$, $b$ in $X$ there are open sets $G$ and $H$ in $s$ satisfying

1. $a \in f^{-1}(G)$, $b \in H$ and $b \not\in f^{-1}(G)$, $a \not\in H$

or

2. $b \in f^{-1}(G)$, $a \in H$ and $a \not\in f^{-1}(G)$, $b \not\in H$.

Now $f^{-1}(G)$ is open in $t$ and hence $f^{-1}(G) = \{a\}$ or $X$. Since $b \not\in f^{-1}(G)$, or $a \not\in f^{-1}(G)$, it follows that $f^{-1}(G) = \{a\}$. So certainly (1) holds. Now $G \neq X$, otherwise $f^{-1}(G) = X$, which is not true. So $G = \{a\}$ or $G = \{b\}$.

**Case (i) $G = \{a\}$.**

Since $b \in H$, $a \not\in H$, it follows that $H = \{b\}$. Thus $s$ is the discrete topology.

Now $f$ is continuous. So $f(a) = a$ and $f(b) = b$ is not possible, because $f^{-1}(H) = f^{-1}(b) = \{b\}$, which is not open in $t$. Since $f^{-1}(G) = \{a\}$ and since $G = \{a\}$, it follows that $f(a) = a$. Then $f(b) = a$ must hold. This contradicts the fact that $f$ is surjective.

**Case (ii) $G = \{b\}$.**

Since $f^{-1}(G) = \{a\}$ so $f(a) = b$. Since $G$ and $H$ are distinct, $H = \{a\}$ or $X$. But $a \not\in H$ implies $H \neq X$. So $H = \{a\}$. Then $f(a) = b$. Since $f$ is bijective, $f(b) = a$. But $f^{-1}(H) = \{b\} \not\in t$, which contradicts the fact that $f$ is continuous. Thus in any case we have a contradiction. We conclude that $(X, t)$ cannot be star-$T_1$.

**NOTE 2.5** Star-$T_1$ axiom is lying between $T_0$ and $T_1$ axioms.

**Star-$T_2$ -spaces**

**DEFINITION 3.1** A topological space $(X, t)$ is said to be star-$T_2$ if there exists a topology $s$ on $X$ and a bijective continuous function $f: (X, t) \to (X, s)$ such that for any two distinct points $a$, $b$ in $X$ there are open sets $G$ and $H$ in $s$ satisfying the following
condition:

\[ a \in f^{-1}(G), \ b \in H, \ f^{-1}(G) \cap H = \emptyset \]

or

\[ b \in f^{-1}(G), \ a \in H, \ f^{-1}(G) \cap H = \emptyset. \]

**NOTE 3.1** If \((X, t)\) is \(T_2\) then it is star-\(T_2\).

For, consider \(s = t\) and \(f\) the identity function.

**NOTE 3.2** A star-\(T_2\) -space may not be \(T_2\).

**EXAMPLE 3.1** Let \(X = \{a, b, c\}\), \(t = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}\), \(s = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}\). Define \(f: (X, t) \rightarrow (X, s)\) by \(f(a) = a, f(b) = c, f(c) = b\). Then \(f\) is bijective and continuous. Now for the points \(a, b\), choose \(G = \{c\}, H = \{a\}\). Then \(b \in f^{-1}(G), a \in H, f^{-1}(G) \cap H = \emptyset\). For the points \(b, c\), choose \(G = \{c\}, H = \{a, c\}\). Then \(b \in f^{-1}(G), c \in H, f^{-1}(G) \cap H = \emptyset\). For the points \(a, c\), choose \(G = \{a, c\}, H = \{c\}\). Then \(a \in f^{-1}(G), c \in H, f^{-1}(G) \cap H = \emptyset\). Hence \((X, t)\) is star-\(T_2\). Also \((X, t)\) is not \(T_2\).

**NOTE 3.3** A star-\(T_2\) -space may not be \(T_1\).

In example 3.1, \((X, t)\) is star-\(T_2\) but \((X, t)\) is not \(T_1\).

**NOTE 3.4** A \(T_1\) -space may not be star-\(T_2\).

**EXAMPLE 3.2** Let \(X\) be an infinite set and \(t\) be the cofinite topology on \(X\). Then \((X, t)\) is \(T_1\). We claim that \((X, t)\) is not star-\(T_2\). If possible, let it be star-\(T_2\). Then there exists a topology \(s\) on \(X\) and a bijective continuous function \(f: (X, t) \rightarrow (X, s)\) such that for any two distinct points \(a, b\) in \(X\) there are open sets \(G\) and \(H\) in \(s\) satisfying the following condition:

\[ (1) \ a \in f^{-1}(G), \ b \in H, \ f^{-1}(G) \cap H = \emptyset \]

or

\[ (2) \ b \in f^{-1}(G), \ a \in H, \ f^{-1}(G) \cap H = \emptyset. \]

Let \(a, b \in X\) and \((1)\) hold. Then \(f^{-1}(G) \in t\). Then \(X - f^{-1}(G)\) is a finite -1 set. Since \(f^{-1}(G) \cap H = \emptyset, H \subset X - f^{-1}(G)\). So \(H\) is a finite set. Since \(-1 H \neq \emptyset\) and \(f^{-1}(H)\) is open and \(f\) is surjective, so \(f^{-1}(H)\) is an infinite set with finite complement. But \(f\) is injective, so \(f^{-1}(H)\) cannot be infinite (since \(H\) is finite). Thus a contradiction arises.

Hence \((1)\) cannot hold. Similarly we can show that \((2)\) cannot hold. Hence \((X, t)\) cannot be star-\(T_2\).

**NOTE 3.5** A star-\(T_2\) -space is star- \(T_1\).
Some New Separation Axioms: A Different Approach

Proof follows from the definitions.

**Star-Regular-spaces**
Recall that a topological space \((X, t)\) is regular if for any point \(a\) in \(X\) and any closed set \(F\) in \((X, t)\) where \(a \notin F\), there exist two open sets \(G\) and \(H\) in \((X, t)\) satisfying \(a \in G\), \(F \subset H\), \(G \cap H = \emptyset\).

We now have the following definitions:

**DEFINITION 4.1** A topological space \((X, t)\) is said to be A-star-regular if there exists a topology \(s\) on \(X\) and a bijective continuous function \(f:\ (X, t) \to (X, s)\) such that for any point \(a\) in \(X\) and any closed set \(F\) in \((X, t)\) where \(a \notin F\), there are open sets \(G\) and \(H\) in \(s\) satisfying the condition that \(a \in f^{-1}(G)\), \(F \subset H\), \(f^{-1}(G) \cap H = \emptyset\).

**DEFINITION 4.2** A topological space \((X, t)\) is said to be B-star-regular if there exists a topology \(s\) on \(X\) and a bijective continuous function \(f:\ (X, t) \to (X, s)\) such that for any point \(a\) in \(X\) and any closed set \(F\) in \((X, t)\) where \(a \notin F\), there are open sets \(G\) and \(H\) in \(s\) satisfying the condition that \(F \subset f^{-1}(G)\), \(a \in H\), \(f^{-1}(G) \cap H = \emptyset\).

**NOTE 4.1** A regular space is A-star-regular.

For, consider \(s = t\) and \(f\), the identity function.

**NOTE 4.2** An A-star-regular space may not be regular.

**EXAMPLE 4.1** Let \(X = \{a, b, c\}\), \(t = \{\emptyset, X, \{a\}, \{a, b\}\}\), \(s = \{\emptyset, X, \{c\}, \{b, c\}\}\).

Then \(c(t) = \{\emptyset, X, \emptyset, \{b, c\}, \{c\}\}\), where \(c(t)\) denotes the set of all closed sub-sets in \((X, t)\). Define \(f:\ (X, t) \to (X, s)\) by \(f(a) = c\), \(f(b) = b\), \(f(c) = a\).

Then \(f\)'s bijective and continuous. Now for the point \(a\) and closed set \(F = \{b, c\}\), choose \(G = \{c\} \in s\), \(H = \{b, c\} \in s\) then \(a \in f^{-1}(G)\), \(F \subset H\), \(f^{-1}(G) \cap H = \emptyset\).

For the point \(b\) and closed set \(F = \{c\}\), choose \(G = \{b, c\}\), \(H = \{c\}\). Hence \((X, t)\) is A-star-regular. Clearly it is not regular since \(\{a\}\) and \(\{b, c\}\) cannot be separated by open sets in \(t\).

**NOTE 4.3** A regular space is B-star-regular.

For, consider \(s = t\) and \(f\), the identity function.

**THEOREM 4.1** A finite B-star-regular space is regular.

**PROOF:** If possible, let \((X, t)\) be a finite B-star regular space which is not regular. Then there exist \(a \in X\), \(F \in C(t)\) such that \(a \notin F\) and for every open set \(G_a\) and every open set \(H_F\), \(a \in G_a\), \(F \subset H_F \Rightarrow G_a \cap H_F \neq \emptyset\). Since \((X, t)\) is B-star-regular, there exists...
a topology $s$ on $X$ and a bijective continuous function $f: (X, t) \to (X, s)$ satisfying the B-star-regularity condition. So there exist $G \in s$, $H \in s$ such that $F \subset f^{-1}(G)$, $a \in H$, $f^{-1}(G) \cap H = \emptyset$.\hspace{1cm}(1)

Now $F$ is not open in $(X, t)$, otherwise $a \in X - F$, $F \subset F$ and $(X-F) \cap F = \emptyset$, a contradiction. Since $F$ is continuous $f^{-1}(G) \subset t$. So $f^{-1}(G) - F \neq \emptyset$. Let $a_1 \in f^{-1}(G) - F$.

$$f^{-1}(G) \cap \text{cl}_t \{a_1\} = \emptyset.$$\hspace{1cm}(2)

This implies $a_1 \notin \text{cl}_t \{a\}$. Now consider the closed set $F \cup \text{cl}_t \{a\}$ in $(X, t)$ where $a_1 \notin F \cup \text{cl}_t \{a\}$. Again by B-star-regularity condition, there exists $G_1 \in s$, $H_1 \in s$ such that $F \cup \text{cl}_t \{a\} \subset f^{-1}(G_1)$, $a_1 \in H_1$, $f^{-1}(G_1) \cap H_1 = \emptyset$.\hspace{1cm}(3)

Now we claim that $F \cup \text{cl}_t \{a\} \cup \text{cl}_t \{a_1\}$ is not open in $(X, t)$. Otherwise, if it is open in $(X, t)$ then $F \cup \text{cl}_t \{a\} \cup \text{cl}_t \{a_1\} \cap F = \emptyset$, a contradiction. Thus $F \cup \text{cl}_t \{a\} \cup \text{cl}_t \{a_1\} \cup \text{cl}_t \{a_2\}$ is a closed set in $(X, t)$. We can now use B-star-regularity condition and note that $F \cup \text{cl}_t \{a\} \cup \text{cl}_t \{a_1\} \cup \text{cl}_t \{a_2\}$ is a closed set in $(X, t)$.

**NOTE 4.4** It follows from above theorem that in example 4.1, the space $(X, t)$ is not B-star-regular. Thus the class of A-star regular spaces is different from the class of B-star-regular spaces.

**THEOREM 4.2** A star-$T_1$ and A-star-regular space is star-$T_2$. 
PROOF. Since \((X, t)\) is star-\(T_1\), there exists a topology \(s\) on \(X\) and a bijective continuous function \(f: (X, t) \to (X, s)\) such that for any two distinct points \(a, b\) in \(X\) there are open sets \(G\) and \(H\) in \(s\) satisfying

\((1)\) \(a \in f^{-1}(G), b \in H\) and \(b \notin f^{-1}(G), a \notin H\)

or

\((2)\) \(b \in f^{-1}(G), a \in H\) and \(a \notin f^{-1}(G), b \notin H\).

Let \(a, b \in X\) and \(a = b\). Suppose that \((1)\) holds. Then \(a \notin X - f^{-1}(G) = F\) say, where \(F\) is closed in \((X, t)\). Since \((X, t)\) is A-star-regular there exists a topology \(\mu\) on \(X\) and a bijective continuous function \(g: (X, t) \to (X, \mu)\) such that for any point \(p\) in \(X\) and any closed set \(V\) in \((X, t)\) where \(p \notin V\), there are open sets \(M\) and \(W\) in \(\mu\) satisfying \(p \in g^{-1}(M), V \subset W, g^{-1}(M) \cap W = \emptyset\).

Now \(a \in X\) and \(F\) is closed in \((X, t)\) and \(a \notin F\). Then there are open sets \(M\) and \(W\) in \(\mu\) satisfying \(a \in g^{-1}(M), F \subset W, g^{-1}(M) \cap W = \emptyset\). -1

Now \(b \notin f^{-1}(G) \Rightarrow b \in F\). So \(a \in g^{-1}(M), b \in W, g^{-1}(M) \cap W = \emptyset\).

Hence if \((1)\) holds then the condition for the space \((X, t)\) to be star-\(T_2\) is satisfied. If \((2)\) holds then it can similarly be shown that the condition for the space \((X, t)\) to be star-\(T_2\) is satisfied. This completes the proof of the theorem.

**THEOREM 4.3** A star-\(T_1\) and B-star-regular space is star-\(T_2\).

PROOF. Since \((X, t)\) is star-\(T_1\), there exists a topology \(s\) on \(X\) and a bijective continuous function \(f: (X, t) \to (X, s)\) such that for any two distinct points \(a, b\) in \(X\) there are open sets \(G\) and \(H\) in \(s\) satisfying

\((1)\) \(a \in f^{-1}(G), b \in H\) and \(b \notin f^{-1}(G), a \notin H\)

or

\((2)\) \(b \in f^{-1}(G), a \in H\) and \(a \notin f^{-1}(G), b \notin H\).

Suppose that \(a\) and \(b\) are two distinct points in \(X\) and let \((1)\) hold. Then \(a \notin X - f^{-1}(G) = F\) say, where \(F\) is closed in \((X, t)\). Since \((X, t)\) is B-star-regular, by the B-star regularity condition, there exists a topology \(\mu\) on \(X\) and a bijective continuous function \(g: (X, t) \to (X, \mu)\) such that there are open sets \(M\) and \(W\) in \(\mu\) satisfying \(F \subset g^{-1}(M), a \in W, g^{-1}(M) \cap W = \emptyset\). -1

Now \(b \notin f^{-1}(G) \Rightarrow b \in F\). Hence We have a topology \(\mu\) -1 on \(X\) and a bijective continuous function \(g: (X, t) \to (X, \mu)\) such that for the points \(a\) and \(b\) there are open sets \(M\) and \(W\) in \(\mu\) satisfying \(b \in g^{-1}(M), a \in W, g^{-1}(M) \cap W = \emptyset\). Hence \((X, t)\) is star-\(T_2\).

**DEFINITION 4.2** A topological space \((X, t)\) is said to be A-star-\(T_3\) (respectively, B-star-\(T_3\)) if it is star-\(T_1\) and A-star-regular (respectively, B-star-regular).
NOTE 4.3 An A-star-T₃ space may not be T₂. In example 4.1, the space (X, t) is star-T₁ and A-star-regular and hence star-T₃. But (X, t) is not T₂.

Star-Normal-spaces

DEFINITION 5.1 A topological space (X, t) is said to be star-normal if there exists a topology s on X and a bijective continuous function f: (X, t) → (X, s) such that for any two disjoint closed sets A and B in (X, t), there are open sets G and H in s satisfying the condition that A ⊂ f⁻¹(G), B ⊂ H, f⁻¹(G) ∩ H = Ø.

NOTE 5.1 It is clear from the definition that a T₁ star-normal space is A-star-regular and B-star-regular.

THEOREM 5.1 Let (X, t) be compact and T₁. Then the following statements are equivalent:

• (X, t) is A-star-regular.
• (X, t) is star-normal.
• (X, t) is B-star-regular.

Proof: We shall prove that (i) ⇒ (ii) ⇒ (iii) ⇒ (ii) ⇒ (i).

Let (i) hold. Let A and B be any two disjoint closed sets in (X, t). Since (X, t) is A-star-regular, there exists a topology s on X and a bijective continuous function f: (X, t) → (X, s) such that for any point a in X and any closed set F in (X, t) where a ∉ F, there are open sets G and H in s satisfying the condition that a ∈ f⁻¹(G), F ⊂ H, f⁻¹(G) ∩ H = Ø.

Then for each a ∈ A, there exist open sets Ga, Ha ∈ s such that a ∈ f⁻¹(Ga), B ⊂ Ha, f⁻¹(Ga) ∩ Ha = Ø. Since f is continuous, {f⁻¹(Ga)}a∈A is an open covering of A in (X, t). Since (X, t) is compact and A is closed in (X, t), A is compact in (X, t). So there exist a₁, a₂,........aₙ ∈ A such that A ⊂ f⁻¹(Ga₁) ∪ f⁻¹(Ga₂) ∪ f⁻¹(Ga₃) ∪ .......f⁻¹(Gan) = f⁻¹(Ga₁ ∪ Ga₂ ∪ Ga₃ ∪ ....... Gan) = f⁻¹(G) say, where G = Ga₁ ∪ Ga₂ ∪ Ga₃ ∪ ....... Gan.

Let H = Hₙ₁ ∩ Hₙ₂ ∩ Hₙ₃ ∩ ....... Hₙₙ. Then B ⊂ H.

A ⊂ f⁻¹(G), f⁻¹(G) ∩ H = Ø. Hence (X, t) is star-normal. So (ii) follows.

Now if (ii) holds then (iii) follows easily.

Let (iii) hold. Then since (X, t) is B-star-regular, let us call the s-topology on X under consideration as B-star-regular-s-topology on X.

We now have the following lemma.

LEMMA 5.1 If (X, t) is B-star-regular then for any two disjoint closed sets A and B in (X, t), A ∩ clₛ B = Ø, and B ∩ clₛ A = Ø, where clₛ B denotes closure of B w. r. t. the B-star regular s-topology under consideration.
PROOF OF LEMMA: Let A and B be any two disjoint closed sets in (X, t). Let \( x \in A \). Then \( x \not\in B \). By the B-star-regularity condition there exist \( G, H \) belonging to the B-star regular s-topology on X such that \( B \subseteq f^{-1}(G), \ x \in H \) and \( f^{-1}(G) \cap H = \emptyset \).

Since \( H \in s \), \( cl_s f^{-1}(G) \cap H = \emptyset \), and \( cl_s B \subseteq cl_s f^{-1}(G) \) implies \( cl_s B \cap H = \emptyset \). Thus \( x \in A \Rightarrow x \not\in cl_s B \). This implies \( A \cap cl_s B = \emptyset \). Similarly we can show that \( B \cap cl_s A = \emptyset \). This completes the proof of the lemma.

Now consider two disjoint closed sets A and B in (X, t). By the above lemma, \( A \cap cl_s B = \emptyset \), and where \( cl_s B \) denotes closure of B w. r. t. the B-star regular s-topology under consideration.

Let \( x \in cl_s B \). Then \( x \not\in A \). Then there exist \( G_x, H_x \in s \) such that \( A \subseteq f^{-1}(G_x), \ x \in H_x \) and \( f^{-1}(G_x) \cap H_x = \emptyset \).

Consider \( U = \{H_x: x \in cl_s B\} \). Since (X, t) is compact and f is continuous and surjective, (X, s) is compact. Then \( cl_s B \) is compact in s. Now U is an open covering of \( cl_s B \) in s. So there exist \( x_1, x_2, \ldots, x_n \) in \( cl_s B \) such that \( cl_s B \subseteq H_{x_1} \cup H_{x_2} \cup \ldots \cup H_{x_n} \).

Let \( M = H_{x_1} \cup H_{x_2} \cup \ldots \cup H_{x_n} \) and \( P = G_{x_1} \cap G_{x_2} \cap \ldots \cap G_{x_n} \). Then \( M \in s \) and \( P \in s \).

Now \( f^{-1}(G_{x_1} \cap G_{x_2} \cap \ldots \cap G_{x_n}) = f^{-1}(G_{x_1}) \cap f^{-1}(G_{x_2}) \cap \ldots \cap f^{-1}(G_{x_n}) \) and \( A \subseteq f^{-1}(P) \).

This implies \( A \subseteq f^{-1}(P) \) and \( B \subseteq M \). Also \( f^{-1}(P) \cap M = \emptyset \).

Hence \((X, t) \) is star-normal. So (ii) holds.

Now if (ii) holds then (i) follows easily. This completes the proof of the theorem.

The following problem can be raised which remains unsolved in this paper.

Problem
To construct an example of an infinite B-star-regular space which is not regular.

References