Global Journal of Mathematical Sciences: Theory and Practical.
ISSN 0974-3200 Volume 3, Number 3 (2011), pp. 209-215

http://www.irphouse.com

Uniqueness of Entire and Meromorphic Functions with their Nonlinear Differential Polynomials

Harina P. Waghamore and Tanuja A.

*Department of Mathematics, Central College Campus,
Bangalore University, Bangalore-560 001, India.
E-mail: pree.tam@rediffmail.com, a.tanuja1@gmail.com*

Abstract

In this paper, we study the uniqueness of two entire and meromorphic functions with their nonlinear differential polynomials. We consider the case for some general differential polynomials $[f^n P(f)\ f^{'}]$ where $P(f)$ is a polynomial which generalize and improve previous results of Fang and Hong[1] and Lahiri and Mandal [7].

2000 Mathematics Subject Classification: Primary 30D35.

Keywords and phrases: Entire Functions, Meromorphic Functions, Nonlinear Differential polynomials, Uniqueness.

Introduction

In this paper, we use the standard notations and terms in the value distribution theory [12]. For any nonconstant meromorphic function $f(z)$ on the complex plane C, we denote by $\mathrm{S(r,f)}$ any quantity satisfying $\mathrm{S(r,f)} = \mathrm{o}\{T(r,f)\}$, as $r \to +\infty$, except possibly for a set of r of finite linear measures. A meromorphic function $a(z)$ is called a small function with respect to $\mathrm{f}(z)$ if $\mathrm{T(r,a)} = \mathrm{S(r,f)}$. Let $\mathrm{S}(f)$ be the set of meromorphic function in the complex plane C which are small functions with respect to f. Set $\mathrm{E}(a(z),\mathrm{f}) = \{z: f(z) - a(z) = 0\}, \mathrm{a(z)} \in \mathrm{S(f)}$, where a zero point with multiplicity m is counted m times in the set. If these zero points are only counted once, then we denote the set by $\bar{E}$ (a, f). Let k be $a(z)$ a positive integer. Set $E_{k)}(a(z)$, f) $=\{z: f(z) - a(z) = 0, \exists i, 1 \le i \le k, s.t., f^{(i)}(z) - a^{(i)}(z) \neq 0\}$, where a zero point with multiplicity m is counted m times in the set.

Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, $a(z) \in S(f) \cap S(g)$.

If $\mathrm{E}(a(z),\mathrm{f}) = \mathrm{E}(a(z),\mathrm{g})$, then we say that f(z) and g(z) share the value $a(z)$ CM, especially, we say that f(z) and g(z) have the same fixed points when $a(z) = z$.

If $\bar{E}(\mathrm{a},\mathrm{f}) = \bar{E}(\mathrm{a},\mathrm{g})$, then we say that f(z) and g(z) share the $a(z)$ IM. If $E_{k)}(a(z),\mathrm{f}) = E_{k)}(a(z),\mathrm{g})$, we say that f(z) − a and g(z) − a have same zeros with the same multiplicities $\leq$ k.

Moreover, we also use the following notations.

We denote by $N_{k)}(r,f)$ the counting function for poles of $f(z)$ with multiplicities $\leq k$, and by $\bar{N}_{k)}(r,f)$ the corresponding one for which the multiplicity is not counted. Let $N_{(k}(r,f)$ be the counting function for poles of $f(z)$ with multiplicities $\geq k$, and let $\bar{N}_{(k}(r,f)$ be the corresponding one for which the multiplicity is not counted. Set $N_k(r,f) = \bar{N}(r,f) + \bar{N}_{(2}(r,f) + \cdots + \bar{N}_{(k}(r,f)$.

Similarly, We have the notations

$$N_{k)}\left(r,\frac{1}{f}\right), \bar{N}_{k)}\left(r,\frac{1}{f}\right), N_{(k}\left(r,\frac{1}{f}\right), \bar{N}_{(k}\left(r,\frac{1}{f}\right), N_k\left(r,\frac{1}{f}\right).$$

Let f(z) and g(z) be two nonconstant meromorphic functions and $\bar{E}(1,\mathrm{f}) = \bar{\mathrm{E}}(1,\mathrm{g})$.

We denote by $\bar{N}_L(r,\frac{1}{(f-1)})$ the counting function for 1-points of both f(z) and g(z) about which f(z) has larger multiplicity than g(z), with multiplicity is not being counted, and denote by $N_{1]}(r,\frac{1}{(f-1)})$ the counting function for common simple 1-points of both $f(z)$ and g(z) where multiplicity is not counted. Similarly, we have the notation $\bar{N}_L\left(r,\frac{1}{(g-1)}\right)$.

During the last few years, a considerable amount of work is being done on the uniqueness problem concerning differential polynomials (cf. [1, 4, 5, 6]). Recently, Fang and Hong[1] proved the following result.

Theorem A: Let f and g be two transcendental entire function and n($\geq$ 11) be an positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g$ share 1CM, then $f \equiv g$.

In 2005, Lahiri and Mandal [7] proved the following two theorems.

Theorem B: Let f and g be two transcendental entire functions and n($\geq$ 10) be an positive integer. If $E_{2)}(1;f^n(f-1)f') = E_{2)}(1;g^n(g-1)g')$, then $f \equiv g$.

Theorem C: Let f and g be two transcendental meromorphic functions such that $\Theta(\infty;f) + \Theta(\infty;g) > \frac{4}{n+1}$ and let $n(\geq 17)$ be an positive integer. If $E_{2)}(1;f^n(f-1)f') = E_{2)}(1;g^n(g-1)g')$, then $f \equiv g$.

Naturally we can ask whether there exists a corresponding unicity theorem to Theorem B and Theorem C for $[f^nP(f)\ f']$ where $P(f)$ is a polynomial. In this paper, we give a positive answer to above question and prove the following two theorems.

Uniqueness of Nonlinear Differential Polynomials

Theorem 1.1: Let f and g be two transcendental meromorphic functions. Let $P(f) = a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0, (a_m \neq 0)$, and $a_i (i = 0, 1, \ldots, m)$ is the first nonzero coefficient from the right, and n, m, k be a positive integer with $n(> m + 10)$, $k \geq 3$. If $E_{k)}(1; f^n P(f) f') = E_{k)}(1; g^n P(g) g')$, then $f \equiv g$.

Theorem 1.2: Let f and g be two transcendental entire functions. Let $P(f) = a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0, (a_m \neq 0)$, and $a_i (i = 0, 1, \ldots, m)$ is the first nonzero coefficient from the right, and n, m, k be a positive integer with $n(> m + 6)$, $k \geq 3$. If $E_{k)}(1; f^n P(f) f') = E_{k)}(1; g^n P(g) g')$, then $f \equiv g$.

Lemmas

In this section, we present some lemmas which are needed in the sequel.

Lemma 2.1: ([8, 10]) Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + \cdots + a_n f^n$, where $a_0, a_1, \ldots, a_n$ are constants and $a_n \neq 0$. Then

$$T\big(r, P(f)\big) = nT(r, f) + S(r, f).$$

Lemma 2.2: ([11]) Let f be a nonconstant meromorphic function. Then

$$N\big(r, 0; f^{(k)}\big) \leq k\bar{N}(r, \infty; f) + N(r, 0; f) + S(r, f).$$

Lemma 2.3: Let f and g be two nonconstant meromorphic functions. Then $f^n P(f) f' g^n P(g) g' \neq 1$ where $n + m(\geq 6)$ is an positive integer.

Proof: Let

(2.1) $$f^n P(f) f' g^n P(g) g' \equiv 1$$

Let z_0 be a 1-point of f with multiplicity $p(\geq 1)$. Then z_0 is a pole of g with multiplicity $q(\geq 1)$ such that $np + p - 1 = nq + q + mq + 1$,
i.e.,

(2.2) $$mq + 2 = (n + 1)(p - q)$$

From (2.2) we get $q \geq \frac{n-1}{m}$ and again from (2.2) we obtain

$$p \geq \frac{1}{n+1}\left[\frac{(n+m+1)(n-1)}{m} + 2\right] = \frac{n+m-1}{m}.$$

Let z_1 be a zero of P(f) with multiplicity $p_1(\geq 1)$. Then z_1 is a pole of g with multiplicity $q_1(\geq 1)$, say. So from (2.1) we get

$$2p_1 - 1 = (n + m + 1)q + 1$$
$$\geq (n + m + 2)$$

i.e.,

$$p_1 \geq \frac{(n+m+3)}{2}$$

Since a pole of f is either a zero of g^nP(g) or a zero of g'. we have

$$\overline{N}(r,\infty;f) \leq \overline{N}(r,0;g) + \overline{N}(r,0;g^m) + \overline{N}_0(r,0;g') + S(r,f) + S(r,g)$$
$$\leq \frac{m}{n+m-1} N(r,0;g) + \frac{2}{n+m+3} N(r,0;g^m) + \overline{N}_0(r,0;g')$$
$$+S(r,f) + S(r,g)$$
$$\leq \left(\frac{m}{n+m-1} + \frac{2m}{n+m+3}\right) T(r,g) + \overline{N}_0(r,0;g') + S(r,f) + S(r,g).$$

Where $\overline{N}_0(r,0;g')$ denotes the reduced counting function of those zeros of g' which are not the zeros of gP(g).

As $P(f) = a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0$ where $a_m, a_{m-1}, \ldots, a_0$ are m distinct complex numbers. Then by second fundamental theorem of Nevanlinna we get

$$mT(r,f) \leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \sum_{j=1}^{m} \overline{N}(r,a_j;f) - \overline{N}_0(r,0;f') + S(r,f)$$
$$\leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,a;f^m) - \overline{N}_0(r,0;f') + S(r,f)$$

(2.3) $$\leq \left(\frac{m}{n+m-1} + \frac{2m}{n+m+3}\right)\{T(r,g) + T(r,f)\} + \overline{N}_0(r,0;g')$$
$$-\overline{N}_0(r,0;f') + S(r,f) + S(r,g).$$

Similarly, we have

$$mT(r,g) \leq \left(\frac{m}{n+m-1} + \frac{2m}{n+m+3}\right)\{T(r,g) + T(r,f)\} + \overline{N}_0(r,0;f')$$

(2.4) $$-\overline{N}_0(r,0;g') + S(r,f) + S(r,g).$$

Adding (2.3) and (2.4) we obtain

$$\left(1 - \frac{2}{n+m-1} - \frac{4}{n+m+3}\right)\{T(r,g) + T(r,f)\} \leq S(r,f) + S(r,g).$$

which is a contradiction. This proves the Lemma.

Lemma 2.4: ([2]) Let f and g be two nonconstant meromorphic functions, and let k be two positive integer. If $E_{k)}(1,\mathrm{f}) = \mathrm{E}_{\mathrm{k})}(1,\mathrm{g})$, then one of the following cases must occur:

$$T(r,f) + T(r,g) \leq N_2(r,\infty;f) + N_2(r,0;f) + N_2(r,\infty;g) + N_2(r,0;g)$$
$$+\overline{N}(r,1;f) + \overline{N}(r,1;g) - N_{1]}(r,1;f) + \overline{N}(r,1;f| \geq k+1)$$

$+\overline{N}(r,1;g| \geq k+1)+S(r,f)+S(r,g).$

ii) $f=\frac{(b+1)g+(a-b-1)}{bg+(a-b)}$, where $a \neq 0, b$ are two constants.

Lemma 2.5: ([3]) Let f and g be two nonconstant meromorphic functions. If f and g share 1IM, then one of the following cases must occur:

i. $T(r,f)+T(r,g) \leq 2[N_2(r,\infty;f)+N_2(r,0;f)+N_2(r,\infty;g)+N_2(r,0;g)]$
$+3\overline{N}_L(r,1;f)+3\overline{N}_L(r,1;g)+S(r,f)+S(r,g).$

ii. $f=\frac{(b+1)g+(a-b-1)}{bg+(a-b)}$, where $a \neq 0, b$ are two constants.

Lemma 2.6: Let f and g be two transcendental meromorphic functions, n($> m+6$) be positive integer, and let $F = f^nP(f)f'$ and $G = g^nP(g)g'$. If

(2.5) $$F=\frac{(b+1)G+(a-b-1)}{bG+(a-b)},$$

where $a \neq 0, b$ are two constants, then $f \equiv g$.

Proof: Using the same argument as in [9], we obtain Lemma 2.6 .

Lemma 2.7: Let f and g be two transcendental meromorphic function and

$$F_1=f^{n+1}\left[\frac{a_m}{m+n+1}f^m+\frac{a_{m-1}}{m+n}f^{m-1}+\cdots+\frac{a_0}{n+1}\right]$$
$$G_1=g^{n+1}\left[\frac{a_m}{m+n+1}g^m+\frac{a_{m-1}}{m+n}g^{m-1}+\cdots+\frac{a_0}{n+1}\right]$$

where n($> m+2$) is an integer. Then $F \equiv G$ implies that $F_1 \equiv G_1$.

Proof: Let $F \equiv G$, then $F_1 \equiv G_1+c$ where c is a constant. Let $c \neq 0$. Then by second fundamental theorem we get

$$T(r,F_1) \leq \overline{N}(r,\infty;F_1)+\overline{N}(r,0;F_1)+\overline{N}(r,c;F_1)+S(r,F_1)$$
$$\leq \overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\overline{N}\left(r,\frac{a_m}{m+n+1};f^m\right)$$
$$+\overline{N}(r,0;g)+\overline{N}\left(r,\frac{a_m}{m+n+1};g^m\right)+S(r,f)$$
$$\leq 2T(r,f)+mT(r,f)+T(r,g)+mT(r,g)+S(r,f).$$

Hence we get

(2.6) $$(m+n+1)T(r,f) \leq (2+m)T(r,f)+(m+1)T(r,g)+S(r,f).$$

Similarly, we have

(2.7) $$(m+n+1)T(r,g) \leq (2+m)T(r,g)+(m+1)T(r,f)+S(r,g).$$

Adding (2.6) and (2.7) we obtain

$(m+n+1)\{T(r,f)+T(r,g)\} \leq (3+2m)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)$.
i.e.,
$(n-m-2)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g)$.

which is a contradiction. So $c=0$ and the Lemma is proved.

Proofs of the Theorems

Proof of Theorem 1.1: Let $F=f^nP(f)f'$and G= $g^nP(g)g$.

Since $k \geq 3$, we have

$\overline{N}(r,1;F)+\overline{N}(r,1;G)-N_{1]}(r,1;F)+\overline{N}(r,1;F| \geq k+1)+\overline{N}(r,1;G| \geq k+1)$

$\leq \frac{1}{2}N(r,1;F)+\frac{1}{2}N(r,1;G) \leq \frac{1}{2}T(r,F)+\frac{1}{2}T(r,G)+S(r,f)+S(r,g)$.

Then (i) in Lemma 2.4 becomes

$T(r,F)+T(r,G) \leq 2\{N_2(r,\infty;F)+N_2(r,0;F)+N_2(r,\infty;G)+N_2(r,0;G)\}$
$+S(r,f)+S(r,g)$.

By the definition of F, G we have

(3.1) $N_2(r,\infty;F)+N_2(r,0;F) \leq 2\overline{N}(r,\infty;f)+2\overline{N}(r,0;f)+$
$N(r,c_1;f)+\cdots+N(r,c_m;f)\ +N(r,0;f')$.

Similarly, we obtain

(3.2) $N_2(r,\infty;G)+N_2(r,0;G) \leq 2\overline{N}(r,\infty;g)+2\overline{N}(r,0;g)+$
$N(r,c_1;g)+\cdots+N(r,c_m;g)\ +N(r,0;g')$.

By Lemma 2.2 and (2.6), (2.7) and (3.1), we get

$T(r,F)+T(r,G) \leq 4\overline{N}(r,\infty;f)+4\overline{N}(r,0;f)+2N(r,c_1;f)+\cdots+2N(r,c_m;f)$

$+2N(r,0;f')+4\overline{N}(r,\infty;g)+4\overline{N}(r,0;g)+2N(r,c_1;g)+\cdots+$

$+2N(r,c_m;g)+2N(r,0;g')+S(r,f)+S(r,g)$.

Then

$(n+m+2)\{T(r,f)+T(r,g)\}$
$\leq (12+2m)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)$.

By n $>$ 10 + m, we get a contradiction.

Hence F and G satisfy (ii) in Lemma 2.4.

By Lemma 2.3 and Lemma 2.7, we get f ≡ g. This completes the proof.

Proof of Theorem 1.2: Since f and g are entire functions we have $\overline{N}(r,\infty;f) = \overline{N}(r,\infty;g) = 0$. Proceeding as in the proof of Theorem 1.1 we can easily prove Theorem 1.2.

Acknowledgement

First Author wishes to thank "Bangalore University Internal Research Funding, BUIRF" for the financial support.

References

[1] M.-L.Fang and W.Hong, A Unicity theorem for entire functions concerning differential polynomials, Indian.J.Pure Appl.Math. 32(2001), no.9, 1343-1348.

[2] C.-Y.Fang and M.-L.Fang, Uniqueness of meromorphic functions and differential polynomials, Computers and Mathematics with Applications, vol.44, no.5-6, pp.607-617, 2002.

[3] M. Fang and H. Guo, On unique range sets for meromorphic or entire functions , Acta Mathematica Sinica, vol. 14, pp.569-576, 1998.

[4] I.Lahiri, Uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points, Ann. Polon. Math. 71 (1999), no.2, 113-128.

[5] I.Lahiri, Linear differential polynomials sharing the same 1-points with weight two, Ann. Polon. Math. 79 (2002), no.2, 157-170.

[6] W.C.Lin, Uniqueness of differential polynomials and a problem of Lahiri, Pure. Appl. Math. (Xi'an) 17 (2001), no.2, 104-110 (Chinese).

[7] I.Lahiri and N.Mandal, Uniqueness of nonlinear differential polynomials sharing simple and double 1-points, International journal of Mathematics and Mathematical Science, 12 (2005) 1933-1942.

[8] Mohonko, A.Z., On the Nevanlinna characteristics of some meromorphic functions, The Theory of Functions, Functional Analysis and Their Applications, Izd. Kharkovsk. Univ. 14 (1971), 83-87.

[9] H.-Y. Xu and T.-B. Cao, Uniqueness of entire or meromorphic functions sharing one value or a function with finite weight, Journal of inequalities in pure and applied mathematics, vol.10, iss.3, art.88, 2009.

[10] Yang, C.C., On deficiencies of differential polynomials II, Math .Z. 125 (1972), 107-112.

[11] Yi.H.X., Uniqueness of meromorphic functions and a question of C.C.Yang, Complex Variables Theory Appl.14 (14) (1990), 169-176.

[12] L.Yang, Value distribution Theory, Springer, Berlin, Germany, 1993.