Uniqueness of Entire and Meromorphic Functions with their Nonlinear Differential Polynomials

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Abstract

In this paper, we study the uniqueness of two entire and meromorphic functions with their nonlinear differential polynomials. We consider the case for some general differential polynomials \([f^n P(f) f']\) where \(P(f)\) is a polynomial which generalize and improve previous results of Fang and Hong [1] and Lahiri and Mandal [7].

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Introduction

In this paper, we use the standard notations and terms in the value distribution theory [12]. For any nonconstant meromorphic function \(f(z)\) on the complex plane \(C\), we denote by \(S(r,f)\) any quantity satisfying \(S(r,f) = o(T(r,f))\), as \(r \to +\infty\), except possibly for a set of \(r\) of finite linear measures. A meromorphic function \(a(z)\) is called a small function with respect to \(f(z)\) if \(T(r,a) = S(r,f)\). Let \(S(f)\) be the set of meromorphic function in the complex plane \(C\) which are small functions with respect to \(f\). Set \(E(a(z),f) = \{z; f(z) - a(z) = 0\}, a(z) \in S(f)\), where a zero point with multiplicity \(m\) is counted \(m\) times in the set. If these zero points are only counted once, then we denote the set by \(E(a,f)\). Let \(k\) be a \(a(z)\) a positive integer. Set \(E_k(a(z),f) = \{z; f(z) - a(z) = 0, \exists i, 1 \leq i \leq k, s.t., f^{(i)}(z) - a^{(i)}(z) \neq 0\}\), where a zero point with multiplicity \(m\) is counted \(m\) times in the set.

Let \(f(z)\) and \(g(z)\) be two transcendental meromorphic functions, \(a(z) \in S(f) \cap S(g)\).
If \( E(a(z), f) = E(a(z), g) \), then we say that \( f(z) \) and \( g(z) \) share the value \( a(z) \) CM, especially, we say that \( f(z) \) and \( g(z) \) have the same fixed points when \( a(z) = z \).

If \( \bar{E}(a, f) = \bar{E}(a, g) \), then we say that \( f(z) \) and \( g(z) \) share the value \( a \) IM. If \( E_k(a(z), f) = E_k(a(z), g) \), we say that \( f(z) - a \) and \( g(z) - a \) have same zeros with the same multiplicities \( \leq k \).

Moreover, we also use the following notations.

Let \( f(z) \) and \( g(z) \) be two nonconstant meromorphic functions and \( \Delta \) (1, \( f \)) = 1, \( g \)).

We denote by \( N_k(r, f) \) the counting function for poles of \( f(z) \) with multiplicities \( \leq k \), and by \( \bar{N}_k(r, f) \) the corresponding one for which the multiplicity is not counted. Let \( N_k(r, f) \) be the counting function for poles of \( f(z) \) with multiplicities \( \geq k \), and let \( \bar{N}_k(r, f) \) be the corresponding one for which the multiplicity is not counted.

\[ N_k \left( r, \frac{1}{f-1} \right), N_k \left( r, 1 \right), N_k \left( r, \frac{1}{f} \right), N_k \left( r, \frac{1}{f} \right). \]

Similarly, We have the notations

\[ N_k \left( r, \frac{1}{f} \right), N_k \left( r, 1 \right), N_k \left( r, \frac{1}{f} \right), N_k \left( r, \frac{1}{f} \right). \]

During the last few years, a considerable amount of work is being done on the uniqueness problem concerning differential polynomials (cf. [1, 4, 5, 6]). Recently, Fang and Hong[1] proved the following result.

**Theorem A:** Let \( f \) and \( g \) be two transcendental entire function and \( n(\geq 11) \) be an positive integer. If \( f^n(f - 1)f' \) and \( g^n(g - 1)g' \) share 1CM, then \( f \equiv g \).

In 2005, Lahiri and Mandal [7] proved the following two theorems.

**Theorem B:** Let \( f \) and \( g \) be two transcendental entire functions and \( n(\geq 10) \) be an positive integer. If \( E_2(1; f^n(f - 1)f') = E_2(1; g^n(g - 1)g') \), then \( f \equiv g \).

**Theorem C:** Let \( f \) and \( g \) be two transcendental meromorphic functions such that \( \Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+1} \) and let \( n(\geq 17) \) be an positive integer. If \( E_2(1; f^n(f - 1)f') = E_2(1; g^n(g - 1)g') \), then \( f \equiv g \).

Naturally we can ask whether there exists a corresponding unicity theorem to Theorem B and Theorem C for \( [f^nP(f) f'] \) where \( P(f) \) is a polynomial. In this paper, we give a positive answer to above question and prove the following two theorems.
Uniqueness of Entire and Meromorphic Functions

Theorem 1.1: Let $f$ and $g$ be two transcendental meromorphic functions. Let

$$P(f) = a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0, \quad (a_m \neq 0),$$

and $a_i (i = 0, 1, \ldots, m)$ is the first nonzero coefficient from the right, and $n, m, k$ be a positive integer with $n (> m + 10), k \geq 3$. If $E_{k,j}(1; f^m P(f) f') = E_{k,j}(1; g^m P(g) g')$, then $f \equiv g$.

Theorem 1.2: Let $f$ and $g$ be two transcendental entire functions. Let

$$P(f) = a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0, \quad (a_m \neq 0),$$

and $a_i (i = 0, 1, \ldots, m)$ is the first nonzero coefficient from the right, and $n, m, k$ be a positive integer with $n (> m + 6), k \geq 3$. If $E_{k,j}(1; f^m P(f) f') = E_{k,j}(1; g^m P(g) g')$, then $f \equiv g$.

Lemmas

In this section, we present some lemmas which are needed in the sequel.

Lemma 2.1: ([8, 10]) Let $f$ be a nonconstant meromorphic function and

$$P(f) = a_0 + a_1 f + \cdots + a_n f^n,$$

where $a_0, a_1, \ldots, a_n$ are constants and $a_n \neq 0$. Then

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 2.2: ([11]) Let $f$ be a nonconstant meromorphic function. Then

$$N(r, 0; f^{(k)}) \leq k \bar{N}(r, \infty; f) + N(r, 0; f) + S(r, f).$$

Lemma 2.3: Let $f$ and $g$ be two nonconstant meromorphic functions. Then

$$f^n P(f) f' g^n P(g) g' \equiv 1$$

where $n + m (\geq 6)$ is an positive integer.

Proof: Let

$$f^n P(f) f' g^n P(g) g' \equiv 1$$

Let $z_0$ be a 1-point of $f$ with multiplicity $p (\geq 1)$. Then $z_0$ is a pole of $g$ with multiplicity $q (\geq 1)$ such that $np + p - 1 = nq + q + mq + 1$, i.e.,

$$mq + 2 = (n + 1)(p - q)$$

From (2.2) we get $q \geq \frac{n-1}{m}$ and again from (2.2) we obtain

$$p \geq \frac{1}{n+1} \left[ \left( \frac{n+m+1}{m} \right)^{n-1} + 2 \right] = \frac{n+m-1}{m}.$$

Let $z_1$ be a zero of $P(f)$ with multiplicity $p_1 (\geq 1)$. Then $z_1$ is a pole of $g$ with multiplicity $q_1 (\geq 1)$, say. So from (2.1) we get

$$2p_1 - 1 = (n + m + 1)q + 1 \geq (n + m + 2)$$
Since a pole of \( f \) is either a zero of \( g^nP(g) \) or a zero of \( g' \), we have
\[
N(r, \infty; f) \leq \bar{N}(r, 0; g) + \bar{N}(r, 0; g^n) + \bar{N}_0(r, 0; g') + S(r, f) + S(r, g)
\]
\[
\leq \frac{m}{n+m-1} N(r, 0; g) + \frac{2}{n+m+3} N(r, 0; g^m) + \bar{N}_0(r, 0; g') + S(r, f) + S(r, g)
\]
\[
+ S(r, f') + S(r, g)
\]
\[
\leq \left( \frac{m}{n+m-1} + \frac{2m}{n+m+3} \right) T(r, g) + \bar{N}_0(r, 0; g') + S(r, f) + S(r, g).
\]

Where \( \bar{N}_0(r, 0; g') \) denotes the reduced counting function of those zeros of \( g' \) which are not the zeros of \( gP(g) \).

As \( P(f) = a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0 \) where \( a_m, a_{m-1}, \ldots, a_0 \) are \( m \) distinct complex numbers. Then by second fundamental theorem of Nevanlinna we get
\[
mT(r, f) \leq \bar{N}(r, \infty; f) + \bar{N}(r, 0; f) + \sum_{j=1}^{m} \bar{N}(r, a_j; f) - \bar{N}_0(r, 0; f') + S(r, f)
\]
\[
\leq \bar{N}(r, 0; f) + \bar{N}(r, \infty; f) + \bar{N}(r, a; f^m) - \bar{N}_0(r, 0; f') + S(r, f)
\]
\[
\leq \left( \frac{m}{n+m-1} + \frac{2m}{n+m+3} \right) \{T(r, g) + T(r, f)\} + \bar{N}_0(r, 0; g') - \bar{N}_0(r, 0; f') + S(r, f) + S(r, g).
\]
(2.3)

Similarly, we have
\[
mT(r, g) \leq \left( \frac{m}{n+m-1} + \frac{2m}{n+m+3} \right) \{T(r, g) + T(r, f)\} + \bar{N}_0(r, 0; f') - \bar{N}_0(r, 0; g') + S(r, f) + S(r, g).
\]
(2.4)

Adding (2.3) and (2.4) we obtain
\[
\left( 1 - \frac{2}{n+m-1} - \frac{4}{n+m+3} \right) \{T(r, g) + T(r, f)\} \leq S(r, f) + S(r, g).
\]

which is a contradiction. This proves the Lemma.

**Lemma 2.4:** ([2]) Let \( f \) and \( g \) be two nonconstant meromorphic functions, and let \( k \) be two positive integer. If \( E_{k_1}(1, f) = E_{k_2}(1, g) \), then one of the following cases must occur:
\[
T(r, f) + T(r, g) \leq N_2(r, \infty; f) + N_2(r, 0; f) + N_2(r, \infty; g) + N_2(r, 0; g)
\]
\[
+ \bar{N}(r, 1; f) + \bar{N}(r, 1; g) - N_1(r, 1; f) + \bar{N}(r, 1; f) \geq k + 1
\]
Uniqueness of Entire and Meromorphic Functions

\[ + \tilde{N}(r, 1; g) \geq k + 1 + S(r, f) + S(r, g). \]

ii) \( f = \frac{(b+1)g+(a-b-1)}{bg+(a-b)}, \) where \( a \neq 0, b \) are two constants.

Lemma 2.5: ([3]) Let \( f \) and \( g \) be two nonconstant meromorphic functions. If \( f \) and \( g \)
share 1IM, then one of the following cases must occur:

i. \( T(r, f) + T(r, g) \leq 2[N_2(r, \infty; f) + N_2(r, 0; f) + N_2(r, \infty; g) +
N_2(r, 0; g)] + 3\tilde{N}_L(r, 1; f) + 3\tilde{N}_L(r, 1; g) + S(r, f) + S(r, g). \)

ii. \( f = \frac{(b+1)g+(a-b-1)}{bg+(a-b)}, \) where \( a \neq 0, b \) are two constants.

Lemma 2.6: Let \( f \) and \( g \) be two transcendental meromorphic functions, \( n(> m+6) \)
be positive integer, and let \( F = f^nP(f)/f' \) and \( G = g^nP(g)/g' \). If

\[ (2.5) \quad F = \frac{(b+1)g+(a-b-1)}{bg+(a-b)}, \]

where \( a \neq 0, b \) are two constants, then \( f \equiv g. \)

Proof: Using the same argument as in [9], we obtain Lemma 2.6.

Lemma 2.7: Let \( f \) and \( g \) be two transcendental meromorphic function and

\[ F_1 = f^{n+1} \left[ \frac{a_m}{m+n+1} f^m + \frac{a_{m-1}}{m+n} f^{m-1} + \ldots \right] + \frac{a_0}{n+1} \]
\[ G_1 = g^{n+1} \left[ \frac{a_m}{m+n+1} g^m + \frac{a_{m-1}}{m+n} g^{m-1} + \ldots \right] + \frac{a_0}{n+1} \]

where \( n(> m + 2) \) is an integer. Then \( F \equiv G \) implies that \( F_1 \equiv G_1. \)

Proof: Let \( F \equiv G \), then \( F_1 \equiv G_1 + c \) where \( c \) is a constant. Let \( c \neq 0 \). Then by second
fundamental theorem we get

\[ T(r, F_1) \leq \tilde{N}(r, \infty; F_1) + \tilde{N}(r, 0; F_1) + \tilde{N}(r, c; F_1) + S(r, F_1) \]
\[ \leq \tilde{N}(r, \infty; f) + \tilde{N}(r, 0; f) + \tilde{N} \left( r, \frac{a_m}{m+n+1}; f^m \right) + \tilde{N}(r, \infty; g) + \tilde{N} \left( r, \frac{a_m}{m+n+1}; g^m \right) + S(r, f) \]
\[ \leq 2T(r, f) + mT(r, f) + T(r, g) + mT(r, g) + S(r, f). \]

Hence we get

\[ (2.6) \quad (m + n + 1)T(r, f) \leq (2 + m)T(r, f) + (m + 1)T(r, g) + S(r, f). \]

Similarly, we have

\[ (2.7) \quad (m + n + 1)T(r, g) \leq (2 + m)T(r, g) + (m + 1)T(r, f) + S(r, g). \]
Adding (2.6) and (2.7) we obtain

\[(m + n + 1)\{T(r, f) + T(r, g)\} \leq (3 + 2m)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g)\]  

i.e.,  
\[(n - m - 2)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g)\]  

which is a contradiction. So \(c = 0\) and the Lemma is proved.

**Proofs of the Theorems**

**Proof of Theorem 1.1:** Let \(F = f^nP(f)f'\) and \(G = g^nP(g)g\).

Since \(k \geq 3\), we have
\[
\overline{N}(r, 1; F) + \overline{N}(r, 1; G) - N_{3}(r, 1; F) + \overline{N}(r, 1; F) \geq k + 1 + \overline{N}(r, 1; G) \geq k + 1
\]
\[
\leq \frac{1}{2}N(r, 1; F) + \frac{1}{2}N(r, 1; G) \leq \frac{1}{2}T(r, F) + \frac{1}{2}T(r, G) + S(r, f) + S(r, g).
\]

Then (i) in Lemma 2.4 becomes
\[
T(r, F) + T(r, G) \leq 2\{N_{2}(r, \infty; F) + N_{2}(r, 0; F) + N_{2}(r, \infty; G) + N_{2}(r, 0; G)\} + S(r, f) + S(r, g).
\]

By the definition of \(F, G\) we have
\[
(3.1) \quad N_{2}(r, \infty; F) + N_{2}(r, 0; F) \leq 2\overline{N}(r, \infty; f) + 2\overline{N}(r, 0; f) + N(r, c_{1}; f) + \cdots + N(r, c_{m}; f) + N(r, 0; f').
\]

Similarly, we obtain
\[
(3.2) \quad N_{2}(r, \infty; G) + N_{2}(r, 0; G) \leq 2\overline{N}(r, \infty; g) + 2\overline{N}(r, 0; g) + N(r, c_{1}; g) + \cdots + N(r, c_{m}; g) + N(r, 0; g').
\]

By Lemma 2.2 and (2.6), (2.7) and (3.1), we get
\[
T(r, F) + T(r, G) \leq 4\overline{N}(r, \infty; f) + 4\overline{N}(r, 0; f) + 2N(r, c_{1}; f) + \cdots + 2N(r, c_{m}; f)
\]
\[
+ 2N(r, 0; f') + 4\overline{N}(r, \infty; g) + 4\overline{N}(r, 0; g) + 2N(r, c_{1}; g) + \cdots +
\]
\[
+ 2N(r, c_{m}; g) + 2N(r, 0; g') + S(r, f) + S(r, g).
\]

Then
\[
(n + m + 2)\{T(r, f) + T(r, g)\}
\]
\[
\leq (12 + 2m)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g).
\]

By \(n > 10 + m\), we get a contradiction.

Hence \(F\) and \(G\) satisfy (ii) in Lemma 2.4.

By Lemma 2.3 and Lemma 2.7, we get \(f \equiv g\). This completes the proof.
Proof of Theorem 1.2: Since $f$ and $g$ are entire functions we have $\bar{N}(r, \infty; f) = \bar{N}(r, \infty; g) = 0$. Proceeding as in the proof of Theorem 1.1 we can easily prove Theorem 1.2.

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