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Multiobjective Variational Problems Involving Generalized Higher Order Univex Type I Functions

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Abstract

Generalized V-univex type I functions involving higher order derivatives are introduced in the continuous case. Sufficiency and mixed type duality results are established for multiobjective variational problems under generalized higher order V-univexity type I conditions.

Keywords: Multiobjective programming, Variational problems, Mixed type duality, univex type I functions

AMS Subject Classification: 90C29, 90C32.

Introduction

Convexity plays a vital role in many aspects of mathematical programming. To relax convexity assumptions imposed on the functions involved, various generalized notions have been proposed. One of the useful generalizations is generalized (F,ρ)-convexity introduced by Preda [18], an extension of F-convexity defined by Hanson and Mond [10] and generalized ρ-convexity defined by Vial [19,20].

Hanson and Mond [9] considered a dual formulation for a class of variational problems. Some duality results for a class of differentiable multiobjective variational problems were studied in [4]. Mishra and Mukherjee [13] discussed duality for multiobjective variational problems containing generalized (F,ρ)-convex functions. Mukherjee and Rao [16] considered a mixed type dual for multiobjective variational problem and various duality results were established by relating efficient solutions between this mixed type dual pair. Ahmad and Gulati [2] considered a mixed type duality model for multiobjective variational problems and a number of duality results were established by relating proper efficient solutions between this mixed type dual pair. Husain et al. [11] have studied optimality and duality for multiobjective variational problems involving higher order derivatives.

Bector and Singh [3] introduced B-vex functions. Bhatia and Kumar [5] introduced B-vex functions for variational problems. B-type I functions and generalized B-type I functions were recently introduced by Bhatia and Mehra [6]. Further extension in the form of BF-type I functions were made by Bhatia and Sharma [7] for continuous case. Mishra et al. [14] introduced the class of V-univex type I functions and their generalizations. Khazafi and Rueda [12] extended V-univex type I functions for multiobjective variational programming problems and various sufficiency and mixed type duality results were established under generalized V-univex type I functions.

In this paper, we have introduced higher order univex type I functions their generalizations for continuous case. These functions generalize the class of BF-type I functions [7] and (b,F)-convex functions [17]. Using these concepts optimality and duality results have been established for multiobjective variational problems.

Definitions and Preliminaries

We use the following notations for vector inequalities. For $x,y \in R^n$, we have

$$x \leqq y \text{ iff } x_i \leqq y_i,\ i=1,2,\dots,n,$$

$$x \leq y \text{ iff } x \leqq y \text{ and } x \neq y,$$

$$x < y \text{ iff } x_i < y_i,\ i=1,2,\dots,n.$$

Let I=[a,b] be real interval and $K = \{1, 2, ..., k\}$, $M = \{1, 2, ..., m\}$. Let $\phi : IxR^n xR^n xR^n \to R$ be continuously differentiable function. In order to consider $\phi(t,x(t),\dot{x}(t),\ddot{x}(t))$, where $x(t): I \to R^n$ is twice differentiable with its first and second order derivatives $\dot{x}(t)$ and $\ddot{x}(t)$ respectively.

For notational simplicity, we write $x(t),\dot{x}(t),\ddot{x}(t)$ as $x,\dot{x},\ddot{x}$ respectively, as and when necessary. We denote the partial derivatives of ϕ by ϕ_x, $\phi_{\dot{x}}$, and $\phi_{\ddot{x}}$, where

$$\phi_x = \left[\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \dots, \frac{\partial\phi}{\partial x_n}\right]^T, \quad \phi_{\dot{x}} = \left[\frac{\partial\phi}{\partial \dot{x}_1}, \frac{\partial\phi}{\partial \dot{x}_2}, \dots, \frac{\partial\phi}{\partial \dot{x}_n}\right]^T,$$

$$\phi_{\ddot{x}} = \left[\frac{\partial\phi}{\partial \ddot{x}_1}, \frac{\partial\phi}{\partial \ddot{x}_2}, \dots, \frac{\partial\phi}{\partial \ddot{x}_n}\right]^T.$$

The partial derivatives of other functions used will be written similarly. Let $PS(I,R^n)$ denote the space of all piecewise smooth n-dimensional vector functions x defined on compact subset I of R with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differential operator D is given by

$$y = Dx \;\Leftrightarrow\; x(t) = \alpha + \int_a^b y(s)ds$$

in which α is a given boundary value. There $D = \frac{d}{dt}$ except at discontinuties.

We consider the following multiobjective variational problem:

$$\text{(MOP)} \quad \text{Minimize} \left[\int_a^b \left\{ f^1(t,x(t),\dot{x}(t),\ddot{x}(t)) \right\} dt, ..., \int_a^b \left\{ f^k(t,x(t),\dot{x}(t),\ddot{x}(t)) \right\} dt \right],$$

subject to $x(a) = 0 = x(b)$, (1.1)

$\dot{x}(a) = 0 = \dot{x}(b)$,

$h^j(t,x(t),\dot{x}(t),\ddot{x}(t)) \leqq 0, \quad t \in I, \; j \in M,$ (1.2)

$x(t) \in PS(I,R^n)$, (1.3)

where f_i, $i \in K = \{1, 2, ..., k\}$ and h_j, $j \in M = \{1, 2, ..., m\}$ are assumed to be continuously differentiable functions defined on $IxR^n xR^n xR^n$. Let A is the set of feasible solutions of (MOP). Efficiency is defined in the usual sense as defined in [4].

In relation to (MOP), we introduce the following multiple problems (P_r^0) for each r=1,2,…,k;

$$(P_r^0) \quad \text{Minimize} \left[\int_a^b \left\{ f^r(t,x(t),\dot{x}(t),\ddot{x}(t)) \right\} dt \right],$$

subject to (1.1)-(1.3).

$$\int_a^b \left\{ f^i(t,x(t),\dot{x}(t),\ddot{x}(t)) \right\} dt \;\le\; \int_a^b \left\{ f^i(t,x^0(t),\dot{x}^0(t),\ddot{x}^0(t)) \right\} dt, \; i \in K, \; i \neq r.$$

The following lemma can be proved on the lines of Chankong and Haimes [8].

Lemma 2.1: x^0 is an efficient solution to (MOP) if and only if x^0 is an optimum solution of (P_r^0) for each r=1,2,…,k.

Definition 2.1: A functional $F{:}IxR^n xR^n xR^n xR^n xR^n xR^n xR^n \to R$ is said to be sublinear with respect to the eighth variable if for any $x,x^0 \to R^n$, $\dot{x},\dot{x}^0 \in R^n$, $\ddot{x},\ddot{x}^0 \in R^n$,

$$F\left[t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0;\alpha_1+\alpha_2\right] \le F\left[t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0;\alpha_1\right] + F\left[t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0;\alpha_2\right], \tag{A}$$

for any $\alpha_1, \alpha_2 \in R^n$,

and

$F\left[t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0;\alpha a\right] = \alpha F\left[t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0;a\right]$, for any $\alpha \in R$, $\alpha \geq 0$, $a \in R^n$. (B)

We define the following univex type I functions and their generalizations.

Let us consider a sublinear functional F and the functions $f{:}IxR^n xR^n xR^n \to R^k$, $h: IxR^n xR^n xR^n \to R^m$. We assume that f and h are continuously differentiable functions.

Let $\eta(t,x,x^0){:}IxR^n xR^n \to R^n$, $\phi_0{:}R^k \to R^k$, $\phi_1 R^m \to R^m$, $b_0,b_1{:}PS(I,R^n)\times PS(I,R^n) \to R_+$.

Definition 2.2: A pair (f,h) is said to be V-univex type I at $x^0 \in PS(I,R^n)$, with respect to ϕ_0, ϕ_1, b_0, b_1, η such that for all x∈A, we have

$$b_0(x,x^0)\phi_0\left[\int_a^b f(t,x,\dot{x},\ddot{x})dt - \int_a^b f(t,x^0,\dot{x}^0,\ddot{x}^0)dt\right]$$
$$\geqq \int_a^b F\left[\begin{array}{l} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(f_x(t,x^0,\dot{x}^0,\ddot{x}^0)\text{-}(D\eta^T)f_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)+(D^2\eta^T)f_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0))\end{array}\right]dt\,, \tag{2.1}$$

$$- b_1(x,x^0)\phi_1\int_a^b h(t,x^0,\dot{x}^0,\ddot{x}^0)dt$$
$$\geqq \int_a^b F\left[\begin{array}{l} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(h_x(t,x^0,\dot{x}^0,\ddot{x}^0)\text{-}(D\eta^T)h_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)+(D^2\eta^T)h_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0))\end{array}\right]dt. \tag{2.2}$$

If (2.1) is satisfied as a strict inequality then we say that a pair (f,h) is semi-strictly V-univex type I at x^0 with respect to ϕ_0, ϕ_1, b_0, b_1, η.

Remark

1. When ϕ_0, ϕ_1=1 and $D^2\eta$=0, the concept of generalized univex-type I is the same as that of BF-type I in Ref. 7.
2. When ϕ_0, ϕ_1=1, $\eta(x,x^0)$=1 and $D^2\eta$=0, the same concept appeared in the definition of (b,F)-convex functions in Ref. 17.

Definition 2.3: A pair (f,h) is said to be weakly V-strictly pseudoquasi univex type I at $x^0 \in PS(I,R^n)$, with respect to ϕ_0, ϕ_1, b_0, b_1, η such that for all x∈A, we have

$$\phi_0 \int_a^b f(t,x,\dot{x},\ddot{x})dt \;\le\; \phi_0 \int_a^b f(t,x^0,\dot{x}^0,\ddot{x}^0)dt$$

$$\Rightarrow\; b_0(x,x^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(f_x(t,x^0,\dot{x}^0,\ddot{x}^0)\text{-}(D\eta^T)f_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)+(D^2\eta^T)f_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0))\end{array}\right]dt < 0\,,$$

$$-\,\phi_1\int_a^b h(t,x^0,\dot{x}^0,\ddot{x}^0)dt \leqq 0$$

$$\Rightarrow\; b_1(x,x^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(h_x(t,x^0,\dot{x}^0,\ddot{x}^0)\text{-}(D\eta^T)h_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)+ (D^2\eta^T)h_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0))\end{array}\right]dt \leqq 0.$$

Definition 2.4: A pair (f,h) is said to be strongly V-pseudoquasi univex type I at $x^0 \in PS(I,R^n)$, with respect to ϕ_0, ϕ_1, b_0, b_1, η such that for all $x \in A$, we have

$$\phi_0 \int_a^b f(t,x,\dot{x},\ddot{x})dt \;\le\; \phi_0 \int_a^b f(t,x^0,\dot{x}^0,\ddot{x}^0)dt$$

$$\Rightarrow\; b_0(x,x^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(f_x(t,x^0,\dot{x}^0,\ddot{x}^0)\text{-}(D\eta^T)f_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)+(D^2\eta^T)f_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0))\end{array}\right]dt \le 0\,,$$

$$-\,\phi_1\int_a^b h(t,x^0,u^0,\dot{x}^0,\ddot{x}^0)dt \leqq 0$$

$$\Rightarrow\; b_1(x,x^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(h_x(t,x^0,\dot{x}^0,\ddot{x}^0)\text{-}(D\eta^T)h_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)+ (D^2\eta^T)h_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0))\end{array}\right]dt \leqq 0.$$

Definition 2.5: A pair (f,h) is said to be weakly V-strictly pseudo univex type I at $x^0 \in PS(I,R^n)$, with respect to ϕ_0, ϕ_1, b_0, b_1, η such that for all $x \in A$, we have

$$\phi_0 \int_a^b f(t,x,\dot{x},\ddot{x})dt \;\le\; \phi_0 \int_a^b f(t,x^0,\dot{x}^0,\ddot{x}^0)dt$$

$$\Rightarrow\; b_0(x,x^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(f_x(t,x^0,\dot{x}^0,\ddot{x}^0)\text{-}(D\eta^T)f_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)+(D^2\eta^T)f_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0))\end{array}\right]dt < 0\,,$$

$$-\,\phi_1\int_a^b h(t,x^0,u^0,\dot{x}^0,\ddot{x}^0)dt \leqq 0$$

$$\Rightarrow\; b_1(x,x^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(h_x(t,x^0,\dot{x}^0,\ddot{x}^0)\text{-}(D\eta^T)h_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)+(D^2\eta^T)h_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0))\end{array}\right]dt < 0.$$

Sufficient Conditions

In this section, we establish various sufficient optimality conditions for (MOP) under generalized V-univexity type I conditions.

Theorem 3.1: Assume that $x^0 \in A$ is a feasible solution for (MOP) and assume that there exists $\lambda^0 \in R^k$, $\lambda^0 \geq 0$, $\beta^0 \in PS(I, R^m_+)$ such that the following relations hold for all $t \in I$:

$$(\lambda^{0T} f_x(t,x^0,\dot{x}^0,\ddot{x}^0) + \beta^0(t)^T h_x(t,x^0,\dot{x}^0,\ddot{x}^0)) - D(\lambda^{0T} f_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0) + \beta^0(t)^T h_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)) + D^2(\lambda^{0T} f_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0) + \beta^0(t)^T h_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)) = 0, \tag{3.1}$$

$$\beta^0(t)^T h(t,x^0,\dot{x}^0,\ddot{x}^0) = 0, \tag{3.2}$$

$$\beta^0(t) \geqq 0, \ t \in I. \tag{3.3}$$

Further, assume that $(f, \beta^0(t)^T h)$ is strongly V-pseudoquasi univex type I at x^0 with respect to functions ϕ_0, ϕ_1, b_0, b_1, η with $b_1(x,x^0) > 0$ for all $x \in A$. Moreover, suppose that $\phi_1(0) \geqq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$. Then x^0 is an efficient solution for (MOP).

Proof: If x^0 is not an efficient solution for (MOP), then there exists $x \in A$ such that

$$\int_a^b f(t,x,\dot{x},\ddot{x})dt \leq \int_a^b f(t,x^0,\dot{x}^0,\ddot{x}^0)dt.$$

From (3.2), we have

$$\int_a^b \beta^0(t)^T h(t,x^0,\dot{x}^0,\ddot{x}^0)dt = 0.$$

Using $\phi_1(0) \geqq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, we get

$$\phi_0\left[\int_a^b f(t,x,\dot{x},\ddot{x})dt - \int_a^b f(t,x^0,\dot{x}^0,\ddot{x}^0)dt\right] \leq 0, \tag{3.4}$$

$$-\phi_1\left[\int_a^b \beta^0(t)^t h(t,x^0,\dot{x}^0,\ddot{x}^0)dt\right] \leqq 0. \tag{3.5}$$

Since $(f, \beta^0(t)^T h)$ is strongly V-pseudoquasi univex type I at x^0 with respect to ϕ_0, ϕ_1, b_0, b_1, η,

$$b_0(x,x^0)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(f_x(t,x^0,\dot{x}^0,\ddot{x}^0) - (D\eta^T)f_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0) + (D^2\eta^T)f_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)) \end{bmatrix}dt \leq 0,$$

$$b_1(x,x^0)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(\beta^0(t)^T h_x(t,x^0,\dot{x}^0,\ddot{x}^0)-(D\eta^T)\beta^0(t)^T h_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0) \\ +(D^2\eta^T)\beta^0(t)^T h_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)) \end{bmatrix} dt \leqq 0.$$

Since $b_1(x,x^0) > 0$, and $\lambda^0 > 0$, we get

$$b_0(x,x^0)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(\lambda^{0T} f_x(t,x^0,\dot{x}^0,\ddot{x}^0) - (D\eta^T)\lambda^{0T} f_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0) \\ + (D^2\eta^T)\lambda^{0T} f_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)) \end{bmatrix} dt < 0, \tag{3.6}$$

$$\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(\beta^0(t)^T h_x(t,x^0,\dot{x}^0,\ddot{x}^0) - (D\eta^T)\beta^0(t)^T h_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0) \\ + (D^2\eta^T)\beta^0(t)^T h_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)) \end{bmatrix} dt \leqq 0. \tag{3.7}$$

Since $b_0(x,x^0) \geqq 0$, it follows that

$$b_0(x,x^0)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(\beta^0(t)^T h_x(t,x^0,\dot{x}^0,\ddot{x}^0) - (D\eta^T)\beta^0(t)^T h_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0) \\ + (D^2\eta^T)\beta^0(t)^T h_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)) \end{bmatrix} dt \leqq 0. \tag{3.8}$$

Adding (3.6) and (3.8), we get

$$b_0(x,x^0)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T((\lambda^{0T} f_x(t,x^0,\dot{x}^0,\ddot{x}^0) + (\beta^0(t)^T h_x(t,x^0,\dot{x}^0,\ddot{x}^0)) \\ - (D\eta^T)(\lambda^{0T} f_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}) + (\beta^0(t)^T h_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)) \\ + (D^2\eta^T)(\lambda^{0T} f_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0) + (\beta^0(t)^T h_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0))) \end{bmatrix} dt < 0,$$

which contradicts (3.1). Hence x^0 is an efficient solution for (MOP) and it completes the proof.

In the next theorem, we replace strongly V-pseudoquasi univex type I by weakly V-pseudoquasi univex type I of $(f,\beta^0(t)^T h)$.

Theorem 3.2: Assume that $x^0 \in A$ is a feasible solution for (MOP) and there exists $\lambda^0 \in R^k$, $\lambda^0 \geq 0$, $\beta^0 \in PS(I,R_+^m)$ such that (3.1)-(3.3) of theorem 3.1 are satisfied.

Further, assume that $(f,\beta^0(t)^T h)$ is weakly V-pseudoquasi univex type I at x^0 with respect to functions ϕ_0, ϕ_1, b_0, b_1, η with $b_1(x,x^0)>0$ for all $x \in A$. Suppose that $\phi_1(0) \geqq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$. Then x^0 is an efficient solution for (MOP).

Proof: If x^0 is not an efficient solution for (MOP), then there exists $x \in A$ such that

$$\int_a^b f(t,x,\dot{x},\ddot{x})dt \leq \int_a^b f(t,x^0,\dot{x}^0,\ddot{x}^0)dt.$$

From (3.2), we have

$$\int_a^b \beta^0(t)^T h(t,x^0,\dot{x}^0,\ddot{x}^0)dt = 0.$$

Using $\phi_1(0) \geqq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, we get

$$\phi_0\left[\int_a^b f(t,x,\dot{x},\ddot{x})dt - \int_a^b f(t,x^0,\dot{x}^0,\ddot{x}^0)dt\right] \leq 0,$$

$$-\phi_1\left[\int_a^b \beta^0(t)^T h(t,x^0,\dot{x}^0,\ddot{x}^0)dt\right] \leqq 0.$$

Since $(f,\beta^0(t)^T h)$ is weakly V-pseudoquasi univex type I at x^0 with respect to $\phi_0,\phi_1, b_0,b_1,\eta$,

$$b_0(x,x^0)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(f_x(t,x^0,\dot{x}^0,\ddot{x}^0)-(D\eta^T)f_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)+(D^2\eta^T)f_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0))\end{bmatrix}dt < 0,$$

$$b_1(x,x^0)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(\beta^0(t)^T h_x(t,x^0,\dot{x}^0,\ddot{x}^0) - (D\eta^T)\beta^0(t)^T h_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0) \\ + (D^2\eta^T)\beta^0(t)^T h_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0))\end{bmatrix}dt \leqq 0.$$

Remaining part of the proof follows on similar lines as that of theorem 3.1.

In the final sufficiency result below, we invoke the weak V-strictly pseudo univex type I of $(f,\beta^0(t)^T h)$.

Theorem 3.3: Assume that $x^0 \in A$ is a feasible solution for (MOP) and there exists $\lambda^0 \in R^k$, $\lambda^0 \geq 0$, $\beta^0 \in PS(I,R_+^m)$ such that (3.1)-(3.3) of theorem 3.1 are satisfied.

Also, assume that $(f,\beta^0(t)^T h)$ is weakly V-strictly pseudo univex type I at x^0 with respect to functions ϕ_0, ϕ_1, b_0, b_1, η with $b_1(x,x^0)>0$ for all $x \in A$. Suppose that $\phi_1(0) \geqq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$. Then x^0 is an efficient solution for (MOP).

Proof: If x^0 is not an efficient solution for (MOP), then there exists $x \in A$ such that

$$\int_a^b f(t,x,\dot{x},\ddot{x})dt \leq \int_a^b f(t,x^0,\dot{x}^0,\ddot{x}^0)dt.$$

From (3.2), we have

$$\int_a^b \beta^0(t)^T h(t,x^0,\dot{x}^0,\ddot{x}^0)dt = 0.$$

Using $\phi_1(0) \geqq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, we get

$$\phi_0\left[\int_a^b f(t,x,\dot{x},\ddot{x})dt - \int_a^b f(t,x^0,\dot{x}^0,\ddot{x}^0)dt\right] \leq 0,$$

$$-\phi_1\left[\int_a^b \beta^0(t)^t h(t,x^0,\dot{x}^0,\ddot{x}^0)dt\right] \leqq 0.$$

Since $(f,\beta^0(t)^T h)$ is weakly V-strictly pseudo univex type I at x^0 with respect to ϕ_0,ϕ_1, b_0, b_1, η,

$$b_0(x,x^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(f_x(t,x^0,\dot{x}^0,\ddot{x}^0) - (D\eta^T)f_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0) \\ + (D^2\eta^T)f_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)) \end{array}\right]dt < 0, \qquad (3.9)$$

$$b_1(x,x^0)\int_a^b F\left[\begin{array}{l} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(\beta^0(t)^T h_x(t,x^0,\dot{x}^0,\ddot{x}^0) - (D\eta^T)\beta^0(t)^T h_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0) \\ + (D^2\eta^T)\beta^0(t)^T h_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)) \end{array}\right]dt < 0. \qquad (3.10)$$

From (3.9) and (3.10), we have

$$\int_a^b F\left[\begin{array}{l} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(f_x(t,x^0,\dot{x}^0,\ddot{x}^0)-(D\eta^T)f_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)+(D^2\eta^T)f_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)) \end{array}\right]dt < 0, \qquad (3.11)$$

$$\int_a^b F\left[\begin{array}{l} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(\beta^0(t)^T h_x(t,x^0,\dot{x}^0,\ddot{x}^0) - (D\eta^T)\beta^0(t)^T h_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0) \\ + (D^2\eta^T)\beta^0(t)^T h_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)) \end{array}\right]dt < 0. \qquad (3.12)$$

(Since $b_0(x,x^0)>0$, $b_0(x,x^0)>0$)

Since $\lambda^0 \geqq 0$, (3.11) gives

$$\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T(\lambda^{0T}f_x(t,x^0,\dot{x}^0,\ddot{x}^0)-(D\eta^T)\lambda^{0T}f_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)+(D^2\eta^T)\lambda^{0T}f_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)) \end{bmatrix}dt \leqq 0. \qquad (3.13)$$

Adding (3.12) and (3.13), we obtain

$$\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},x^0,\dot{x}^0,\ddot{x}^0; \\ \eta^T((\lambda^{0T}f_x(t,x^0,\dot{x}^0,\ddot{x}^0) + (\beta^0(t)^T h_x(t,x^0,\dot{x}^0,\ddot{x}^0)) \\ - (D\eta^T)(\lambda^{0T}f_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}) + (\beta^0(t)^T h_{\dot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0)) \\ + (D^2\eta^T)(\lambda^{0T}f_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0) + (\beta^0(t)^T h_{\ddot{x}}(t,x^0,\dot{x}^0,\ddot{x}^0))) \end{bmatrix}dt < 0,$$

which contradicts (3.1). Hence the result.

Mixed Type Duality

We divide the index set M of the constraint function of the problem (MOP) into two distinct subsets, namely J_1 and J_2 such that $J_1 \cup J_2 = M$, and let e be the vector of R^k whose components are all ones. We consider the following mixed type dual for (MOP):

(XMOP) Maximize $\int_a^b [(f(t,u,\dot{u},\ddot{u}) + \beta_{J_1}(t)^T h^{J_1}(t,u,\dot{u},\ddot{u})e]dt,$

subject to $x(a) = 0 = x(b),$ (4.1)

$\dot{x}(a) = 0 = \dot{x}(b),$

$$\int_a^b \beta_{J_2}^t(t)h^{J_2}(t,u,\dot{u},\ddot{u})dt \geqq 0, \qquad (4.2)$$

$$\lambda^T\left[f_x(t,u(t),\dot{u}(t),\ddot{u}(t)) - Df_{\dot{x}}(t,u(t),\dot{u}(t),\ddot{u}(t)) + D^2 f_{\ddot{x}}(t,u(t),\dot{u}(t),\ddot{u}(t))\right] \\ + \beta(t)^T\left[h_x(t,u(t),\dot{u}(t),\ddot{u}(t)) - Dh_{\dot{x}}(t,u(t),\dot{u}(t),\ddot{u}(t)) + D^2 h_{\ddot{x}}(t,u(t),\dot{u}(t),\ddot{u}(t))\right] = 0, \qquad (4.3)$$

$$\lambda \in R^k,\ \lambda \geqq 0,\ \lambda^t e = 1,\ e = (1,1,..,1) \in R^k,\ \beta(t) \geqq 0,\ t \in I, \qquad (4.4)$$

where $f(t,u,\dot{u},\ddot{u}) = \left[f^1(t,u,\dot{u},\ddot{u}), f^2(t,u,\dot{u},\ddot{u}),...,f^k(t,u,\dot{u},\ddot{u})\right]$.

We note that we get a Mond-Weir [15] type dual for $J_1 = \varnothing$ and a Wolfe [22] type dual for $J_2 = \varnothing$ in (XMOP) respectively.

We prove various duality results for (MOP) and (XMOP) under generalized V-univexity type I conditions.

Theorem 4.1: Let $x \in A$ and $(u,\lambda,\beta(t)) \in B$. Let any of the following conditions holds:

a. $\lambda > 0$, $(f + \beta_{J_1}(t)^T h_{J_1} e, \beta_{J_2}(t)^T h_{J_2})$ is strongly V-pseudoquasi univex type I at u with respect to $\phi_0, \phi_1, b_0, b_1, \eta$ with $b_1(x,u)>0$ for all $x \in A$. Also $a \leq 0 \Rightarrow \phi_0(a) \leq 0$ and $a \geqq 0 \Rightarrow \phi_1(a) \geqq 0$.

b. $(f + \beta_{J_1}(t)^T h_{J_1} e, \beta_{J_2}(t)^T h_{J_2})$ is weakly V-strictly pseudoquasi univex type I at u with respect to ϕ_0, ϕ_1, b_0, b_1, η with $b_1(x,u) > 0$ for all $x \in A$. Also $a \leq 0 \Rightarrow \phi_0(a) \leq 0$ and $a \geqq 0 \Rightarrow \phi_1(a) \geqq 0$.

c. $(f + \beta_{J_1}(t)^T h_{J_1} e, \beta_{J_2}(t)^T h_{J_2})$ is weakly V-strictly pseudo univex type I at u with respect to ϕ_0, ϕ_1, b_0, b_1, η with $b_1(x,u)>0$ for all $x \in A$. Also $a \leq 0 \Rightarrow \phi_0(a) \leq 0$ and $a \geqq 0 \Rightarrow \phi_1(a) \geqq 0$.

Then the following cannot hold:

$$\int_a^b f(t,x,\dot{x},\ddot{x})dt \leq \int_a^b [f(t,u,\dot{u},\ddot{u}) + \{\beta_{J_1}(t)^t h^{J_1}(t,u,\dot{u},\ddot{u})\}e]dt.$$

Proof: Let x be feasible for (MOP) and $(u,\lambda,\beta(t))$ be feasible for (XMOP). Suppose that

$$\int_a^b f(t,x,\dot{x},\ddot{x})dt \leq \int_a^b [f(t,u,\dot{u},\ddot{u}) + \{\beta_{J_1}(t)^T h^{J_1}(t,u,\dot{u},\ddot{u})\}e]dt.$$

Since x is feasible for (MOP) and $(u,\lambda,\beta(t))$ be feasible for (XMOP), we have

$$\int_a^b [f(t,x,\dot{x},\ddot{x}) + \{\beta_{J_1}(t)^T h^{J_1}(t,x,\dot{x},\ddot{x})\}e]dt \leq \int_a^b [f(t,u,\dot{u},\ddot{u}) + \{\beta_{J_1}(t)^T h^{J_1}(t,u,\dot{u},\ddot{u})\}e]dt. \quad (4.5)$$

Using $a \geqq 0 \Rightarrow \phi_1(a) \geqq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$ with (3.1), we get

$$\phi_0\left[\int_a^b [f(t,x,\dot{x},\ddot{x})+\{\beta_{J_1}(t)^T h^{J_1}(t,x,\dot{x},\ddot{x})\}e]dt - \int_a^b [f(t,u,\dot{u},\ddot{u})+\{\beta_{J_1}(t)^T h^{J_1}(t,u,\dot{u},\ddot{u})\}e]dt\right] \leq 0,$$

$$-\phi_1\left[\int_a^b \beta_{J_2}(t)^T h^{J_2}(t,x,\dot{x},\ddot{x})dt\right] \leqq 0.$$

Since $(f + \beta_{J_1}(t)^T h_{J_1} e, \beta_{J_2}(t)^T h_{J_2})$ is strongly V-pseudoquasi univex type I at u with respect to $\phi_0, \phi_1, b_0, b_1, \eta$

$$b_0(x,u)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},u,\dot{u},\ddot{u}; \\ \eta^T((f_u(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1u}(t,u,\dot{u},\ddot{u})) \\ -(D\eta^T)(f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1\dot{u}}(t,u,\dot{u},\ddot{u})) \\ +(D^2\eta^T)(f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1\ddot{u}}(t,u,\dot{u},\ddot{u}))) \end{bmatrix} dt \leq 0,$$

$$b_1(x,u)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},u,\dot{u},\ddot{u}; \\ \eta^T(\beta_{J_2}(t)^T h_{J_2u}(t,u,\dot{u},\ddot{u})) - (D\eta^T)(\beta_{J_2}(t)^T h_{J_2\dot{u}}(t,u,\dot{u},\ddot{u})) \\ +(D^2\eta^T)(\beta_{J_2}(t)^T h_{J_2\ddot{u}}(t,u,\dot{u},\ddot{u})) \end{bmatrix} dt \leqq 0.$$

Since $b_1(x,u) > 0$ and $\lambda > 0,$ we get

$$b_0(x,u)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},u,\dot{u},\ddot{u}; \\ \eta^T((\lambda^T f_u(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1u}(t,u,\dot{u},\ddot{u})) \\ -(D\eta^T)(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1\dot{u}}(t,u,\dot{u},\ddot{u})) \\ +(D^2\eta^T)(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1\ddot{u}}(t,u,\dot{u},\ddot{u}))) \end{bmatrix} dt < 0, \tag{4.6}$$

$$\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},u,\dot{u},\ddot{u}; \\ \eta^T(\beta_{J_2}(t)^T h_{J_2u}(t,u,\dot{u},\ddot{u})) - (D\eta^T)(\beta_{J_2}(t)^T h_{J_2\dot{u}}(t,u,\dot{u},\ddot{u})) \\ +(D^2\eta^T)(\beta_{J_2}(t)^T h_{J_2\ddot{u}}(t,u,\dot{u},\ddot{u})) \end{bmatrix} dt \leqq 0. \tag{4.7}$$

By $b_0(x,u) \geqq 0,$ it follows that

$$b_0(x,u)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},u,\dot{u},\ddot{u}; \\ \eta^T(\beta_{J_2}(t)^T h_{J_2u}(t,u,\dot{u},\ddot{u})) - (D\eta^T)(\beta_{J_2}(t)^T h_{J_2\dot{u}}(t,u,\dot{u},\ddot{u})) \\ +(D^2\eta^T)(\beta_{J_2}(t)^T h_{J_2\ddot{u}}(t,u,\dot{u},\ddot{u})) \end{bmatrix} dt \leqq 0. \tag{4.8}$$

Adding (4.6) and (4,8), we obtain

$$b_0(x,u)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},u,\dot{u},\ddot{u}; \\ \eta^T((\lambda^T f_u(t,u,\dot{u},\ddot{u}) + \beta(t)^T h_u(t,u,\dot{u},\ddot{u})) \\ -(D\eta^T)(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + \beta(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u})) \\ +(D^2\eta^T)(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + \beta(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}))) \end{bmatrix} dt < 0,$$

which contradicts (4.1).

Now, by hypothesis (b) and from (4.2), (4.5), we get

$$\phi_0\left[\int_a^b [f(t,x,\dot{x},\ddot{x}) + \{\beta_{J_1}(t)^T h^{J_1}(t,x,\dot{x},\ddot{x})\}e]dt - \int_a^b [f(t,u,\dot{u},\ddot{u}) + \{\beta_{J_1}(t)^T h^{J_1}(t,u,\dot{u},\ddot{u})\}e]dt\right] \leq 0,$$

$$-\phi_1\left[\int_a^b \beta_{J_2}(t)^T h^{J_2}(t,x,\dot{x},u,\dot{u})dt\right] \leqq 0.$$

Since $(f+\beta_{J_1}(t)^T h_{J_1} e, \beta_{J_2}(t)^T h_{J_2})$ is weak V-strictly pseudoquasi univex type I at u with respect to ϕ_0, ϕ_1, b_0, b_1, η

$$b_0(x,u)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},u,\dot{u},\ddot{u}; \\ \eta^T((\lambda^T f_u(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1 u}(t,u,\dot{u},\ddot{u})) \\ -(D\eta^T)(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1\dot{u}}(t,u,\dot{u},\ddot{u})) \\ +(D^2\eta^T)(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1\ddot{u}}(t,u,\dot{u},\ddot{u}))) \end{bmatrix} dt < 0,$$

$$b_1(x,u)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},u,\dot{u},\ddot{u}; \\ \eta^T(\beta_{J_2}(t)^T h_{J_2 u}(t,u,\dot{u},\ddot{u})) - (D\eta^T)(\beta_{J_2}(t)^T h_{J_2\dot{u}}(t,u,\dot{u},\ddot{u})) \\ +(D^2\eta^T)(\beta_{J_2}(t)^T h_{J_2\ddot{u}}(t,u,\dot{u},\ddot{u})) \end{bmatrix} dt \leqq 0.$$

Since $b_1(x,u) > 0$ and $\lambda \geq 0$, we get

$$b_0(x,u)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},u,\dot{u},\ddot{u}; \\ \eta^T((\lambda^T f_u(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1 u}(t,u,\dot{u},\ddot{u})) \\ -(D\eta^T)(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1\dot{u}}(t,u,\dot{u},\ddot{u})) \\ +(D^2\eta^T)(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1\ddot{u}}(t,u,\dot{u},\ddot{u}))) \end{bmatrix} dt < 0, \tag{4.9}$$

$$\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},u,\dot{u},\ddot{u}; \\ \eta^T(\beta_{J_2}(t)^T h_{J_2 u}(t,u,\dot{u},\ddot{u})) - (D\eta^T)(\beta_{J_2}(t)^T h_{J_2\dot{u}}(t,u,\dot{u},\ddot{u})) \\ +(D^2\eta^T)(\beta_{J_2}(t)^T h_{J_2\ddot{u}}(t,u,\dot{u},\ddot{u})) \end{bmatrix} dt \leqq 0. \tag{4.10}$$

By $b_0(x,u) \geqq 0$, it follows that

$$b_0(x,u)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},u,\dot{u},\ddot{u}; \\ \eta^T(\beta_{J_2}(t)^T h_{J_2 u}(t,u,\dot{u},\ddot{u})) - (D\eta^T)(\beta_{J_2}(t)^T h_{J_2\dot{u}}(t,u,\dot{u},\ddot{u})) \\ +(D^2\eta^T)(\beta_{J_2}(t)^T h_{J_2\ddot{u}}(t,u,\dot{u},\ddot{u})) \end{bmatrix} dt \leqq 0. \tag{4.11}$$

Adding (4.9) and (4.11), we obtain

$$b_0(x,u)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},u,\dot{u},\ddot{u}; \\ \eta^T((\lambda^T f_u(t,u,\dot{u},\ddot{u}) + \beta(t)^T h_u(t,u,\dot{u},\ddot{u})) \\ -(D\eta^T)(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + \beta(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u})) \\ +(D^2\eta^T)(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + \beta(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}))) \end{bmatrix} dt < 0,$$

which contradicts (4.1).

If (c) holds, then from (4.2) and (4.5), we get

$$\phi_0\left[\int_a^b [f(t,x,\dot{x},\ddot{x})+\{\beta_{J_1}(t)^T h^{J_1}(t,x,\dot{x},\ddot{x})\}e]dt-\int_a^b [f(t,u,\dot{u},\ddot{u})+\{\beta_{J_1}(t)^T h^{J_1}(t,u,\dot{u},\ddot{u})\}e]dt\right]\leq 0,$$

$$-\phi_1\left[\int_a^b \beta_{J_2}(t)^T h^{J_2}(t,x,\dot{x},u,\dot{u})dt\right]\leqq 0.$$

Since $(f+\beta_{J_1}(t)^T h_{J_1}e, \beta_{J_2}(t)^T h_{J_2})$ is weakly V-strictly pseudo univex type I at u with respect to ϕ_0, ϕ_1, b_0, b_1, η

$$b_0(x,u)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},u,\dot{u},\ddot{u}; \\ \eta^T((f_u(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1u}(t,u,\dot{u},\ddot{u})) \\ -(D\eta^T)(f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1\dot{u}}(t,u,\dot{u},\ddot{u})) \\ +(D^2\eta^T)(f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1\ddot{u}}(t,u,\dot{u},\ddot{u}))) \end{bmatrix}dt<0, \tag{4.12}$$

$$b_1(x,u)\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},u,\dot{u},\ddot{u}; \\ \eta^T(\beta_{J_2}(t)^T h_{J_2u}(t,u,\dot{u},\ddot{u})) - (D\eta^T)(\beta_{J_2}(t)^T h_{J_2\dot{u}}(t,u,\dot{u},\ddot{u})) \\ +(D^2\eta^T)(\beta_{J_2}(t)^T h_{J_2\ddot{u}}(t,u,\dot{u},\ddot{u})) \end{bmatrix}dt<0. \tag{4.13}$$

From (4.12) and (4.13), we get

$$\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},u,\dot{u},\ddot{u}; \\ \eta^T((f_u(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1u}(t,u,\dot{u},\ddot{u})) \\ -(D\eta^T)(f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1\dot{u}}(t,u,\dot{u},\ddot{u})) \\ +(D^2\eta^T)(f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1\ddot{u}}(t,u,\dot{u},\ddot{u}))) \end{bmatrix}dt<0, \tag{4.14}$$

$$\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},u,\dot{u},\ddot{u}; \\ \eta^T(\beta_{J_2}(t)^T h_{J_2u}(t,u,\dot{u},\ddot{u})) - (D\eta^T)(\beta_{J_2}(t)^T h_{J_2\dot{u}}(t,u,\dot{u},\ddot{u})) \\ +(D^2\eta^T)(\beta_{J_2}(t)^T h_{J_2\ddot{u}}(t,u,\dot{u},\ddot{u})) \end{bmatrix}dt<0. \tag{4.15}$$

(Since $b_0(x,u)>0$, $b_1(x,u)>0$)

Because $\lambda \geq 0$, (4.14) gives

$$\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},u,\dot{u},\ddot{u}; \\ \eta^T((\lambda^T f_u(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1 u}(t,u,\dot{u},\ddot{u})) \\ -(D\eta^T)(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1\dot{u}}(t,u,\dot{u},\ddot{u})) \\ +(D^2\eta^T)(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + e\beta_{J_1}(t)^T h_{J_1\ddot{u}}(t,u,\dot{u},\ddot{u}))) \end{bmatrix} dt < 0, \tag{4.16}$$

Adding (4.15) and (4.16), we get

$$\int_a^b F\begin{bmatrix} t,x,\dot{x},\ddot{x},u,\dot{u},\ddot{u}; \\ \eta^T((\lambda^T f_u(t,u,\dot{u},\ddot{u}) + \beta(t)^T h_u(t,u,\dot{u},\ddot{u})) \\ -(D\eta^T)(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + \beta(t)^T h_{\dot{u}}(t,u,\dot{u},\ddot{u})) \\ +(D^2\eta^T)(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + \beta(t)^T h_{\ddot{u}}(t,u,\dot{u},\ddot{u}))) \end{bmatrix} dt < 0,$$

which contradicts (4.1).

Corollary 4.1: (See [1]) Let $(u^0,\lambda^0,\beta^0(t))$ be a feasible solution for (XMOP). Assume that $\beta^0_{J_1}(t)^T h_{J_1 u}(t,u^0,\dot{u}^0,\ddot{u}^0) = 0$ and assume that u^0 is a feasible for (MOP). If the weak duality theorem 4.1 holds between (MOP) and (XMOP), then u^0 is an efficient solution for (MOP) and $(u^0,\lambda^0,\beta^0(t))$ is an efficient solution for (XMOP).

Necessary optimality conditions for the existence of an external solution for the single objective variational problem subject to both equality and inequality constraints were given by Valentine [21]. Invoking Valentine' [21] results, Hanson and Mond [9] obtained corresponding necessary optimality conditions. Using the relationship between the efficient solution of the problem (MOP) and the optimal solution of the associated scalar control problem, the necessary optimality conditions were derived for the multiobjective variational problems; details can be found in [6]. Fritz John necessary optimality conditions derived in the form of (3.1)-(3.3) of theorem 3.1 with $\lambda^0 \geqq 0,$ lead to Kuhn-Tucker type necessary optimality conditions under additional constraint qualifications.

Theorem 4.2: (Strong Duality): Let x^0 be feasible solution for (MOP) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exists $\lambda^0 \in R^k$, $\lambda^0 \geq 0$, $\lambda^{0t}e = 1$, $\beta^0 \in PS(I,R^m_+)$ such that $(x^0,\lambda^0,\beta^0(t))$ is feasible for (XMOP) with $\beta^0_{J_1}(t)^T h_{J_1 u}(t,u^0,\dot{u}^0,\ddot{u}^0) = 0$.

If also the weak duality theorem 4.1 holds between (MOP) and (XMOP), then $(x^0,\lambda^0,\beta^0(t))$ is an efficient solution for (XMOP).

Proof: Since x^0 is an efficient solution for (MOP) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exists $\lambda^0 \in R^k$, $\lambda^0 \geq 0$, $\lambda^{0t}e = 1$,

$\beta^0 \in \mathrm{PS}(I,R_+^m)$ such that (3.1)-(3.3) of theorem 3.1 hold. Moreover, $x^0 \in A$, hence the feasibility of $(x^0,\lambda^0,\beta^0(t))$ for (XMOP) follows.

Also because weak duality holds between (MOP) and (XMOP), $(x^0,\lambda^0,\beta^0(t))$ is an efficient solution for (XMOP).

If $(x^0,\lambda^0,\beta^0(t))$ is not an efficient solution for (XMOP), then proceeding along the lines similar to those in Corollary 4.1 in [1], we a get a contradiction to weak duality.

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