Global Journal of Mathematical Sciences: Theory and Practical.
ISSN 0974-3200 Volume 3, Number 3 (2011), pp. 285-288

http://www.irphouse.com

On the Box Topology on $[0,\omega+1]^{\omega}$ Box Topology

[1]Nuno C. Freire and [2]M.F. Veiga

[1]DMAT Universidade de Évora Col. Verney,
R. Romão Ramalho 59, 7000 Évora PORTUGAL
And Mailing Address Av. República nº 975 Lote 2,
2º Dto 2775-274 Parede PORTUGAL
E-mail: freirenuno2003@iol.pt
[2]Faculdade de Ciências e Tecnologia Universidade Nova de Lisboa,
Quinta da Torre 2829-516 Monte de Caparica PORTUGAL
E-mail: mfvo@fct.unl.pt

Abstract

This paper is, we obtain the answer to a Conjecture by Mary Ellen Rudin in the affirmative. Namely, we prove that the box topology on the product $[0,\omega+1]^{\omega}$ is normal.

Introduction

In this paper we prove that the product $[0,\omega+1]^{\omega}$ endowed with the box topology is normal, independently of you accept the Continuum Hypothesis or not. According to Mary Ellen Rudin in [3], the question on normality or paracompactness of box products is pertinent and, in this paper, we obtain in Theorem 3.1. that the box topology on $[0,\omega+1]^{\omega}$ is in fact normal, which is Conjecture 17. by the Author in the above paper. We also notice that the space is paracompact if you accept the Continuum Hypothesis, as follows from the result that Mary Ellen Rudin asserts to have obtained jointly with Kunen, that every box product of countably many locally compact, separable metric spaces (or countably many ordinals) is paracompact under that assumption ([3], p. 191).

Preliminairies

Recall that a Hausdorff topological space is said to be normal if each two disjoint closed subsets have disjoint neighborhoods.

Also recall the box topology ([2]) on the product $\Pi_{i\in I} X_i$, where each (X_i, T_i) is a topological space, which is the topology generated by the base determined by the open rectangles $\Pi_{i\in I} O_i, O_i \in T_i$ $(i\in I)$. We refer to [1] concerning the usual topology on the ordinal space $[0,\omega+1]$ (each subset A of $[0,\omega+1]$ such that $\omega\notin A$ turns out to be open, the class of intervals $\{[n,\omega]:n=0,1,2,...\}$ is a base of open neighborhoods of the point ω).

Remark

For X a metric space and any topological space Y homeomorphic to X, the space Y is metrizable.

Proof

In fact it follows easily that, the topology of X being defined through a metric d and $f:X\to Y$ a homeomorphism, the function $d_Y(f(x),f(y))=d(x,y)$ $(x,y\in X)$ is a metric on the space Y defining the topology.

Remark

The metric subspace $E=\{-1,0,1/n:n=1,2,...\}$ of the real line equipped with the usual topology is homeomorphic to $[0,\omega+1]$ through the homeomorphism $h:E\to[0,\omega+1],h(-1)=\omega+1,h(0)=\omega,h(1/n)=n$ $(n=1,2,...)$.

Proof

This follows immediately.

Lemma

If there is a closed continuous surjection $f:X\to Y$ and the topological space X is normal, then also Y is a normal space.

Proof

This follows from Theorem 3.3 in [1], Chap. VII (p. 145).

The Results

Recall that the Continuum Hypothesis consists of the assumption that, there is no cardinal number λ such that $\omega<\lambda<c$, where we denote by ω the cardinal number of the set of all natural numbers, c for the cardinality of the set of real numbers (see [1], Remark 7.9, Chap. II). We do not need the Continuum hypothesis for the

Theorem

The box topology on $[0,\omega+1]^{\omega}$ is normal.

Proof

Consider the topology T on $E=[0,\omega+1]^{\omega+1}=[0,\omega+1]^{\omega}\times[0,\omega+1]$ determined by the subsets O such that p(O) is open in $[0,\omega+1]^{\omega}$ when equipped with the box topology, where $p((\alpha_n))=(\alpha_n)$ $(0\leq n\leq\omega)$ and, either $(\omega,\omega,...)\notin O$ or $[0,\omega+1)\backslash pr_{\omega+1}((O))$ is finite, $pr_{\omega+1}((\alpha_n))=\alpha_{\omega+1}$. (E,T) is Hausdorff, given $(\alpha_n)\neq(\omega,\omega,\omega,...)$ we may consider the disjoint open sets $\{(\alpha_n)\}$ and $[0,\omega+1]^{\omega}\times([0,\omega+1]\backslash\{\alpha_{\omega+1}\})$, and clearly it is normal due to the fact that each closed subset not containing $(\omega,\omega,\omega,...)$ is open. Also the map $p:E\to[0,\omega+1]^{\omega}, p((\alpha_n))=(\alpha_n)$ $(0\leq n\leq\omega)$ is a continuous surjection; moreover, it follows from the definitions that the closures $p(h(C))\subset h(C)$ for each subset C of E, hence the result follows from [1], Theorem 11.4. Chap. III and Lemma 2.3.

Recall that a Hausdorff topological space X is said to be separable if it contains a countable, dense subset, and, if each point has a compact neighborhood, we say that X is locally compact.

Also recall that a family $\{C_t\}:t\in T\}$ of subsets of a Hausdorff topological space X is locally finite if for each point $x\in X$ there exists a neighborhood Vof x such that the set $\{t\in T:V\cap C_t\}\neq\varphi\}$ is finite. $C=\{O_s:s\in S\}$ being an open cover of X, the open cover $B=\{B_\gamma:\gamma\in\Gamma\}$ is an open refinement of C if for each index $\gamma\in\Gamma$ there is some s such that $B_\gamma\subset O_s$ and we say that X is paracompact if every open cover of X has a locally finite open refinement. The box product of countably many locally compact, separable metrizable spaces is paracompact is the result by Kunen and Mary Ellen Rudin that we notice in the Introduction, hence we have the

Theorem

Assuming the Continuum Hypothesis, the box topology on the product $[0,\omega+1]^{\omega}$ is normal, paracompact.

Proof

Following Remark 2.2., $[0,\omega+1]$ is a metrizable space. Also clearly the space is locally compact (it is compact) and separable, and the proposition follows from above.

Acknowledgement

This work was developed in CIMA-UE with finantial support from FCT (Programa TOCTI-FEDER).

Acknowledgement
This work was partially supported by Financiamento Base 2009 ISFL-1-297 from FCT/MCTES/PT.

References

[1] DUGUNDJI, JAMES Topology Allyn and Bacon, Inc Boston, London, Sydney, Toronto (1966)
[2] MUNKRES, JAMES R. Topology Prentice Hall, Inc. (2000)
[3] RUDIN, MARY ELLEN Some Conjectures (183-193) in Open Problems in Topology, J. van Mill, George M. Reed Editors, North Holland Amsterdam, New York, Oxford, Tokyo (1990).