Global Journal of Mathematical Sciences: Theory and Practical.
ISSN 0974-3200 Volume 3, Number 4 (2011), pp. 377-382
© International Research Publication House
http://www.irphouse.com

Dual Integral Eqoations Involving a Generalised Function-II

Archana Lala

Director, Department of Mathematics
Rishi Chadha Vishvs Girls Institute of Technology, Ghaziabad, U.P., India
E-mail: archanalala2006@yahoo.com

Introduction

In the analysis of mixed boundary value problems, we often encounter pairs of dual integral equation of the form.

(1.1) $$\int_0^\infty k_1(x, u)\, A(u)\, du = \lambda(x),\ 0 \le x < 1$$

(1.2) $$\int_0^\infty k_2(x, u)\, A(u)\, du = \mu(x),\ x > 1$$

Where k_1 and k_2 are the Kernels defined over the whole x-u plane and the function λ (x) and $\mu(x)$ are defined on [0, 1] and $(1, \infty)$ respectively.

Various problems of this type, have been considered by many authors taking the kernels k_i as Bessel function J_υ (x) classical polynomials and generalised hypergeometric functions like G and H etc.

Recently the classical polynomials have been generalised in different ways.

Hence, it is worth considering the solution of dual integral equations which involve such generalised functions as Kernels.

In the present paper we propose to consider the following pair of dual integral equations.

(1.3) $$\int_0^\infty (xy)^{a_1}\, e^{-(xy)^r} F_n^r(xy; a_1; k_1; 1)\, f(y)\, dy = h(x);\ \ 0 \le x < 1$$

(1.4) $$\int_0^\infty (xy)^{a_2}\, e^{-(xy)^r} F_n^r(xy; a_2; k_2; 1)\, f(y)\, dy = g(x);\ \ 1 \le x < \infty$$

where h(x) and g(x) are known, f(x) is to be determined, and $F_n^r(x, a, k, p)$ are gerneralised functions of Chatterjea [1], defined by Rodrigue's Formula.

(1.5) $$F_n^r(x, a, k, p) = x^{-a} e^{px^r} D^n \left[x^{a+kn} e^{-px^r} \right], \ D \equiv \frac{d}{dx}$$

k, r, a and p are parameters.

To solve the equation (1.5) and (1.4) we shall use the theory of fractional Integration operators and Mellin transform to reduce the pair of equations into a single integral equation and finally we shall invert it.

Mellin Transform and Fractional Integration Operatos

The Mellin transform f* (s) of a function f(x) is defined by relation [6].

(2.1) $$f^*(s) = M[f(x)] = \int_0^\infty f(x)\, x^{s-1}\, dx,$$

where $s=\sigma+i\tau$ is a complex veriable.

The inverse Mellin transform of f*(s) is f(x) and is given by

(2.2) $$M^{-1}\left[f^*(s)\right] = f(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} f^*(s)\, x^{-s} ds,$$

If k*(s) and f*(s) denote the Mellin transform of k(x) and f(x) respectively, then [6]

(2.3) $$M\left[\int_0^\infty K(xy)\, f(y)\, dy : s\right] = k^*(s)\, f^*(1-s)$$

Thus from (2.3) we have

(2.4) $$\int_0^\infty K(xy)\, f(y)\, dy = M^{-1}\left[k^*(s)\, f^*(1-s)\ ; x\right]$$

$$= \frac{1}{2\pi i}\int_L k^*(s) f^*(1-s) x^{-s} ds;$$

where L is the suitable counter.

Now take

(2.5) $$k_i(x) = x^{a_i+k_i n} e^{-x^r} F_n^r(x, a; k_i; 1), \ i = 1,2\text{-------}$$

then from Erdelyi [2]

$$k_i^*(s) = \frac{\Gamma s\, \Gamma\left(\frac{1}{r}(s-n+a_i+k_i n)\right)}{\Gamma(s-n)} \tag{2.6}$$

Hence by use of (2.4), (1.3) and (1.4) can be written as

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma s\, \Gamma\left(\frac{1}{r}(s-n+a_1+k_1 n)\right)}{\Gamma(s-n)} f^*(1-s)\, x^{-s} ds = h(x);\ 0 \le x < 1 \tag{2.7}$$

and

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma s\, \Gamma\left(\frac{1}{r}(s-n+a_2+k_2 n)\right)}{\Gamma(s-n)} f^*(1-s)\, x^{-s} ds = g(x);\ 1 \le x < \infty \tag{2.8}$$

Fractional integral operators, that we shall use are, given below [3]

$$\tau(\alpha;\beta;r;w(x)) = \frac{r x^{-r\alpha+r-\beta-1}}{\Gamma(\alpha)} \int_0^x (x^r - v^r)^{\alpha-1}\, v^{\beta} w(v)dv\ ; \tag{2.9}$$

and

$$R(\alpha;\beta;r;w(x)) = \frac{r x^{\beta}}{\Gamma(\alpha)} \int_x^\infty (v^r - x^r)^{\alpha-1}\, v^{-\beta-r\alpha+r-1}\, w(v)dv. \tag{2.10}$$

For r=1, these operators reduce to those studied by Kober

Solution of the problem

We operate (2.8) by (2.10), choosing $\alpha = \frac{1}{r}(a_2-a_1+k_2n-k_1n)$ and $\beta = a_1+k_1n-n$, so that (2.8) changes to

$$\frac{1}{2r\pi i}\int_L \frac{\Gamma s\, \Gamma\left(\frac{1}{r}(s-n+a_1+k_1 n)\right)}{\Gamma(s-n)} x^{-s} f^*(1-s)ds = \frac{r x^{a_1+k_1n-n}}{\Gamma\left(\frac{1}{r}(a_2-a_1+k_2n-k_1n)\right)} \tag{3.1}$$

$$\int_x^\infty (v^r - x^r)^{\alpha-1}\, v^{-\beta-r\alpha+r-1}\, g(v)\, dv,$$

Now, put

$$(3.2) \qquad t(x) = \begin{cases} h(x), \ 0 \le x < 1 \\ \dfrac{r x^{a_1 + k_1 n - n}}{\Gamma\left(\frac{1}{r}(a_2 - a_1 + k_2 n - k_1 n)\right)} \displaystyle\int_x^\infty (v^r - x^r)^{\alpha - 1} v^{-\beta - r\alpha + r - 1} g(v)\, dv; \ 1 \le x < \infty \end{cases}$$

Then from (2.7), (3.1) and (3.2), we get.

$$(3.3) \qquad \frac{1}{2r\pi i} \int_L \frac{\Gamma(s)\,\Gamma\left(\frac{1}{r}(s - n + a_1 + k_1 n)\right)}{\Gamma(s-n)} x^{-s} f^*(1-s)\, ds = t(x),$$

again using (2.4) and (2.6), (3.5) becomes.

$$(3.4) \qquad \int_0^\infty k_1(xy)\ f(y)\, dy = t(x); \ 0 \le x < \infty$$

Thus the pair of dual integral equations (1.1) and (1.2) have been reduced to single integral equation (3.4).

By Mellin transform, (3.3) can be written as

$$(3.5) \qquad K^*(s)\, f^*(1-s) = T^*(s)$$

$$(3.6) \qquad \text{where } K^*(s) = \frac{\Gamma(s)\,\Gamma\left(\frac{1}{r}(s - n + a_1 + k_1 n)\right)}{\Gamma(s-n)}$$

and T*(s) =is the Mellin transform of t(x)

Now replacing s by 1-s, in (3.5) we get

$$(3.7) \qquad f^*(s) = L^*(s)\, T^*(1-s)$$

$$(3.8) \qquad \text{where } L^*(s) = \frac{1}{k^*(1-s)} = \frac{\Gamma(1-s-n)}{\Gamma(1-s)\,\Gamma\left(\frac{1}{r}(1 - s - n + a_1 + k_1 n)\right)}$$

Now taking inverse Mellin transform of

$$(3.9) \qquad f(x) = \int_0^\infty L(xy))\ t(y)\, dy$$

Where

$$\text{(3.10)} \qquad L(x)=H^{1,0}_{2,1}\left[x\left|\begin{matrix}(1,1)\ ((\frac{1}{r}(1\text{-}n\text{+}a_1\text{+}k_1 n),\frac{1}{r})\\(1\text{-}n,1)\end{matrix}\right.\right]$$

and we get

$$\text{(3.11)} \qquad f(x)=\frac{1}{r}\int_0^\infty\left[H^{1,0}_{2,1}\left(x\left|\begin{matrix}(1,1)\ (\frac{1}{r}(1\text{-}n\text{+}a_1\text{+}k_1 n),\frac{1}{r})\\(1\text{-}n,1)\end{matrix}\right.\right)t(y)\right]dy,$$

where t(y) is given by (3.2) and $H^{m,n}_{p+n,q+m}$ are fox's H- Function defined by, [5]

$$\text{(3.12)} \qquad H^{m,n}_{p+n,q+m}\left[x\left|\begin{matrix}(a_1,\alpha_1)\text{-----------}(a_{n+p},\alpha_{n+p})\\(b_1,\beta_1)\text{-----------}(b_{m+q},\beta_{m+q})\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_L \theta(s)x^s ds,$$

Where $\omega = \sqrt{-1}$, $x \neq 0$ is a complex veriable and

$$H^{m,n}_{p+n,q+m}\left[x\left|\begin{matrix}(a_1,\alpha_1)\text{-----------}(a_{n+p},\alpha_{n+p})\\(b_1,\beta_1)\text{-----------}(b_{m+q},\beta_{m+q})\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_L \theta(s)x^s ds,$$

m, n, p and q are non - negative integers;

α j (j = 1---------n+p) and βj (j = 1---------- m + q).

Thus we have proved the following.

Theorem: If f(x) is an unknown function satisfying (2.7) and (2.8), where h(x) and g(x) are some known function, then is given by (3.11)

References

[1] Chatterjea S.K. (1966):- Some operational formulas connected with a function defined by generalized Rodrigue's formula; Acta Math, 17, 3-4, 379-385.

[2] Erdeli A. atc. (1954):- Tables of integral transforms; vol.1; MC. graw hill, New York.

[3] Erdely 1 A (1950-51):- On some functional transformation Univ, Politec. Torino Rend.Sem. Mat.10, 217.-234.

[4] Kober, H.(1940):-On fractional integrals and derivative, Quart. J. Math Oxford Ser.11, 193, -211.

[5] Shrivastava, H.M, Gupta K.C. and Goyal S.P. (1982):-The H function of one and two variable with application; South Asian Publishers, New Delhi, India.
[6] Sneddon, I, N.(1974):- The use of Integral Transform; Tata-Mc-Grew H. Pusl. Co. Ltd. New Delhi, India.