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Fixed Point Theorems in Dislocated Quasi D^*-Metric Spaces

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Abstract

In this paper, we introduce the concept of dislocated quasi D*-metric space and prove some coincidence and fixed point theorems in it.

Keywords: Dislocated quasi D^*-metric, coincidence point, fixed point.

Mathematics Subject Classification: 47H10, 54H25.

Introduction and Preliminaries

In 2005 F.M. Zeyada, G.H. Hassan and M.A. Ahmed [4] established various definitions of dislocated quasi-metric space. C.T. Aage and J.N.Salunke [3] and A. Isufati [1] proved fixed point theorems for a single map and a pair of mappings in dislocated metric spaces.

Dhage [2] introduced the concept of D – metric spaces and proved several fixed point theorems in it. Unfortunately almost all theorems are not valid (Refer [6]).

Recently Sedghi et. al. [5] introduced the concept of D^*-metric spaces and proved some common fixed point theorems. Using D^*-metric concept, we introduced the dislocated quasi D^*-metric on X and prove some fixed and coincidence point theorems.

Definition 1.1: Let X be a non empty set and $D^*: X \times X \times X \rightarrow [0, \infty)$ be a function satisfying

(D_1^*) : $D^*(x,y,z) = 0$ implies $x = y = z$,

(D_2^*) : $D^*(x,y,z) \le D^*(a,y,z) + D^*(x,a,a)$ $\forall$ x, y, z, a $\in$ X.

(D_3^*) : $D^*(x,y,y) = D^*(y,x,x)$ $\forall$ x, y $\in$ X.

Then D^* is called a dislocated quasi D^*-metric on X.
If further, D^* satisfies
(D_4^*) : $D^*(x,y,z) = D^*(y,z,x) = \ldots\ldots\ldots$ (symmetry in all variables)

Then D^* is called a dislocated D^*-metric on X.

Definition 1.2: A sequence $\{x_n\}$ in dislocated quasi D^*-metric space (X, D^*) is called Cauchy if for given $\epsilon > 0$, there exists $n_0 \in N$ such that $n,m \ge n_0$ implies $D^*(x_m,x_n,x_n) < \epsilon$ or $D^*(x_n,x_m,x_m) < \epsilon$.

Definition 1.3: A sequence $\{x_n\}$ in dislocated quasi D^*-metric space (X, D^*) converges to x $\in$ X if

$$\lim_{n\to\infty} D^*(x_n, x, x) = 0 \quad (or) \qquad \lim_{n\to\infty} D^*(x, x_n, x) = 0 \qquad (or) \qquad \lim_{n\to\infty} D^*(x, x, x_n) = 0$$

In this case, we say that x is a dislocated quasi-limit of $\{x_n\}$.

Lemma 1.4: In dislocated quasi D^*-metric space (X, D^*), the dislocated quasi – limit of a sequence is unique.

Proof: Suppose x and y are dislocated quasi – limits of $\{x_n\}$ in X.
Now $0 \le D^*(y,x,x) \le D^*(x_n,x,x) + D^*(y,x_n,x_n)$ from (D_2^*)
$= D^*(x_n,x,x) + D^*(x_n,y,y)$ from (D_3^*)
$\to 0$ as $n \to \infty$.

Hence $D^*(y,x,x) = 0$ which implies that x = y.
Now we give our main results.

The Main Results

Theorem 2.1: Let (X, D^*) be a complete dislocated quasi D^*-metric space and T: X→X be a continuous mapping satisfying

$$(2.1.1)\; D^*(Tx,Ty,Tz) \le \alpha \frac{\left[1 + D^*(x,Tx,z)\right]}{\left[1 + D^*(x,y,z)\right]} D^*(y,Ty,Tz) + \beta D^*(x,y,z)$$

for all $x, y, z \in X$, *where* $\alpha \ge 0, \beta \ge 0$ *with* $\alpha + \beta < 1$. *Then T has unique fixed point in* X.

Proof: Let $x_0 \in X$.

Define $x_{n+1} = Tx_n$, n = 0,1,2,3,………..
If $x_{n+1} = x_n$ for some n, then x_n is a fixed point of T.
Assume that $x_{n+1} \neq x_n$ for all n.

$$D^*(x_n, x_{n+1}, x_{n+1}) = D^*(Tx_{n-1}, Tx_n, Tx_n)$$

$$\leq \alpha \frac{\left[1 + D^*(x_{n-1}, x_n, x_n)\right]}{\left[1 + D^*(x_{n-1}, x_n, x_n)\right]} D^*(x_n, x_{n+1}, x_{n+1}) + \beta D^*(x_{n-1}, x_n, x_n).$$

Thus

$$D^*(x_n, x_{n+1}, x_{n+1}) \leq \frac{\beta}{1-\alpha} D^*(x_{n-1}, x_n, x_n).$$

Now from $\left(D_3^*\right)$, we have

$$D^*(x_{n+1}, x_n, x_n) \leq \lambda D^*(x_n, x_{n-1}, x_{n-1}), \textit{where}\ \lambda = \frac{\beta}{1-\alpha} < 1.$$

Continuing this way, we get

$$D^*(x_{n+1}, x_n, x_n) \leq \lambda^n D^*(x_1, x_0, x_0).$$

Now for m > n, using (D_2^*) repeatedly, we get

$$D^*(x_m, x_n, x_n) \leq D^*(x_{n+1}, x_n, x_n) + D^*(x_{n+2}, x_{n+1}, x_{n+1}) + \ldots\ldots + D^*(x_m, x_{m-1}, x_{m-1})$$

$$\leq (\lambda^n + \lambda^{n+1} + \ldots\ldots\ldots + \lambda^{m-1}) D^*(x_1, x_0, x_0)$$

$$\leq \frac{\lambda^n}{1-\lambda} D^*(x_1, x_0, x_0)$$

$\to 0$ as $n \to \infty$, $m \to \infty$.

Hence $\{x_n\}$ is Cauchy. Since (X, D^*) is a complete dislocated quasi D^*-metric space, there exists $u \in X$ such that $\{x_n\}$ converges to u.
Since T is continuous, we have

$$Tu = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = u.$$

Thus u is a fixed point of T.

Uniqueness: Let x be a fixed point of T.
Then

$$D^*(x,x,x) = D^*(Tx,Tx,Tx)$$

$$\leq \alpha \frac{\left[1+D^*(x,x,x)\right]}{\left[1+D^*(x,x,x)\right]} D^*(x,x,x) + \beta D^*(x,x,x)$$

$= (\alpha+\beta)\, D^*(x,x,x)$

Since $0 \leq \alpha+\beta < 1$, We have $D^*(x,x,x) = 0$.

Thus if x is a fixed point of T, then $D^*(x,x,x) = 0$.

Let x and y be fixed points of T.

Then $D^*(x,x,x) = 0 = D^*(y,y,y)$. Now

$D^*(x,y,y) = D^*(Tx,Ty,Ty)$

$$\leq \alpha \frac{\left[1+D^*(x,x,y)\right]}{\left[1+D^*(x,y,y)\right]} D^*(y,y,y) + \beta D^*(x,y,y)$$

$= \beta\, D^*(x,y,y)$.

Since $0 \leq \beta < 1$, we have $D^*(x,y,y) = 0$.Hence x = y.

Thus the fixed point of T is unique.

Now we give a coincidence point theorem for four mappings in dislocated D^*-metric spaces.

Theorem 2.2: Let (X,D^*) be a complete dislocated D^*-metric space. Let A,B,S,T: $X \to X$ be D^*-continuous mapping satisfying
(2.2.1) AS = SA, BT = TB,
(2.2.2) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$,
(2.2.3) $D^*(Ax,By,z) \leq h \max \{D^*(Sx,Ty,z), D^*(Sx,Ax,z), D^*(Ty,By,z)\}$
for all $x,y,z \in X$, $0 \leq h < 1$.
Then (i) A and S or B and T have a coincidence point in X
or

(ii) The pairs (A,S) and (B,T) have a common coincidence point.

Proof: Let $x_0 \in X$.

Define $\{x_n\}$ and $\{y_n\}$ in X such that

$y_{2n} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, n = 0,1,2,3,……….

Suppose $y_{2n} = y_{2n+1}$ for some n.

Then $Tx_{2n+1} = Bx_{2n+1}$. Hence x_{2n+1} is a coincidence point of T and B.

Suppose $y_{2n+1} = y_{2n+2}$ for some n.

Then $Sx_{2n+2} = Ax_{2n+2}$. Hence x_{2n+2} is a coincidence point of S and A.

Assume that $y_n \neq y_{n+1}$ for all n.

Denote $d_n = D^*(y_n,y_{n+1}, y_{n+1})$

$d_{2n} = D^*(y_{2n},y_{2n+1}, y_{2n+1}) = D^*(y_{2n},y_{2n+1}, y_{2n})$ from (D_3^*) *and* (D_4^*)
$= D^*(Ax_{2n},Bx_{2n+1}, y_{2n})$
$\leq h \max \{D^*(y_{2n-1},y_{2n}, y_{2n}), D^*(y_{2n-1},y_{2n}, y_{2n}), D^*(y_{2n},y_{2n+1}, y_{2n})\}$
$\leq h \max \{ d_{2n-1}, d_{2n-1}, d_{2n}\}$ from (D_3^*) *and* (D_4^*)
Thus $d_{2n} \leq h\, d_{2n-1}$.
$d_{2n+1} = D^*(y_{2n+1},y_{2n+2}, y_{2n+2}) = D^*(y_{2n+2},y_{2n+1}, y_{2n+1})$ from (D_3^*)
$= D^*(Ax_{2n+2},Bx_{2n+1}, y_{2n+1})$
$\leq h \max \{D^*(y_{2n+1},y_{2n}, y_{2n+1}), D^*(y_{2n+1},y_{2n+2}, y_{2n+1}), D^*(y_{2n},y_{2n+1}, y_{2n+1})\}$
$\leq h \max \{ d_{2n}, d_{2n+1}, d_{2n}\}$ from (D_3^*) *and* (D_4^*)
Thus $d_{2n+1} \leq h\, d_{2n}$.
Hence $d_n \leq h\, d_{n-1}$ for n = 1,2,3,…….
Hence $d_n \leq h^n d_0 = h^n D^*(y_0,y_1,y_1)$
Now for m > n and using (D_2^*) ,(D_3^*), (D_4^*) *repeatedly we have*
$D^*(y_n,y_n, y_m) \leq D^*(y_n,y_n, y_{n+1}) + D^*(y_{n+1},y_{n+1}, y_{n+2}) +\ldots\ldots + D^*(y_{m-1},y_{m-1}, y_m)$
$= d_n + d_{n+1} + \ldots + d_{m-1}$
$\leq (h^n + h^{n+1} +\ldots\ldots+h^{m-1}) D^*(y_0,y_1,y_1)$
$\leq \dfrac{h^n}{1-h} D^*(y_0,y_1,y_1)$
$\to 0$ as $n \to \infty$, $m \to \infty$.

Thus $\{y_n\}$ is a Cauchy sequence in the complete dislocated D^*-metric space X. Hence there exists $u \in X$ such that $y_n \to u$.

Clearly the sub sequences $\{Ax_{2n}\} \to u$, $\{Bx_{2n+1}\} \to u$, $\{Tx_{2n+1}\} \to u$ and $\{Sx_{2n+2}\} \to u$.

Since AS = SA and A and S are continuous, we have

$$Au = \lim_{n\to\infty} ASx_{2n} = \lim_{n\to\infty} SAx_{2n} = Su.$$

Since BT = TB and B and T are continuous, we have

$$Bu = \lim_{n\to\infty} BTx_{2n+1} = \lim_{n\to\infty} TBx_{2n+1} = Tu.$$

Thus u is a common coincidence point of the pairs (A,S) and (B,T).

Theorem 2.3: Let (X,D^*) be a complete dislocated D^*-metric space. Let A,B: X→X be D^*-continuous mapping satisfying

$D^*(Ax,By,z) \leq h \max \{D^*(x,y,z), D^*(x,Ax,z), D^*(y,By,z)\}$

for all $x,y,z \in X$ and $0 \leq h < 1$.

Then either A or B a fixed point or A and B have a unique common fixed point.

Proof: Putting S = T = I (Identity map) in Theorem 2.2., we have either A or B has a fixed point or A and B have a common fixed point.

Suppose u and v are two common fixed points of A and B.
$D^*(u,u,v) = D^*(Au,Bu,v)$
$\leq h \max \{D^*(u,u,v), D^*(u,u,v), D^*(u,u,v)\}$
$= h D^*(u,u,v)$.

Since $0 \leq h < 1$, we have that $D^*(u,u,v) = 0$.
Hence u = v.
Thus A and B have a unique common fixed point.

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