Fixed Point Theorems in Dislocated Quasi D*-Metric Spaces

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Abstract

In this paper, we introduce the concept of dislocated quasi D*-metric space and prove some coincidence and fixed point theorems in it.

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Introduction and Preliminaries


Dhage [2] introduced the concept of D – metric spaces and proved several fixed point theorems in it. Unfortunately almost all theorems are not valid (Refer [6]).

Recently Sedghi et al. [5] introduced the concept of D*-metric spaces and proved some common fixed point theorems. Using D*-metric concept, we introduced the dislocated quasi D*-metric on X and prove some fixed and coincidence point theorems.

Definition 1.1: Let X be a non empty set and $D^*: X \times X \times X \rightarrow [0, \infty)$ be a function satisfying
(\(D_1^*\)) \(D^* (x,y,z) = 0\) implies \(x = y = z\),

(\(D_2^*\)) \(D^* (x,y,z) \leq D^* (a,y,z) + D^* (x,a,a)\) \(\forall x, y, z, a \in X\).

(\(D_3^*\)) \(D^* (x,y,y) = D^* (y,x,x)\) \(\forall x, y \in X\).

Then \(D^*\) is called a dislocated quasi \(D^*\)-metric on \(X\).

If further, \(D^*\) satisfies

(\(D_4^*\)) \(D^* (x,y,z) = D^* (y,z,x) = \ldots \ldots\) (symmetry in all variables)

Then \(D^*\) is called a dislocated \(D^*\)-metric on \(X\).

**Definition 1.2:** A sequence \(\{x_n\}\) in dislocated quasi \(D^*\)-metric space \((X, D^*)\) is called Cauchy if for given \(\varepsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that \(n,m \geq n_0\) implies \(D^* (x_m,x_n,x_n) < \varepsilon\) or \(D^* (x_n,x_m,x_m) < \varepsilon\).

**Definition 1.3:** A sequence \(\{x_n\}\) in dislocated quasi \(D^*\)-metric space \((X, D^*)\) converges to \(x \in X\) if

\[
\lim_{n \to \infty} D^* (x_n, x, x) = 0 \ (or) \ \lim_{n \to \infty} D^* (x, x_n, x) = 0 \ (or) \ \lim_{n \to \infty} D^* (x, x, x_n) = 0
\]

In this case, we say that \(x\) is a dislocated quasi-limit of \(\{x_n\}\).

**Lemma 1.4:** In dislocated quasi \(D^*\)-metric space \((X, D^*)\), the dislocated quasi – limit of a sequence is unique.

**Proof:** Suppose \(x\) and \(y\) are dislocated quasi – limits of \(\{x_n\}\) in \(X\).

Now \(0 \leq D^* (y,x,x) \leq D^* (x_n,x, x) + D^* (y,x_n,x_n)\) from \((D_2^*)\)

\[
= D^* (x_n,x,x) + D^* (x_n,y,y)
\]

\[
\to 0 \ \text{as} \ n \to \infty.
\]

Hence \(D^* (y,x,x) = 0\) which implies that \(x = y\).

Now we give our main results.

**The Main Results**

**Theorem 2.1:** Let \((X, D^*)\) be a complete dislocated quasi \(D^*\)-metric space and \(T: X \to X\) be a continuous mapping satisfying

\[
(2.1.1) \ D^* (Tx, Ty, Tz) \leq \alpha \left[ \frac{1 + D^* (x, Tx, z)}{1 + D^* (x, y, z)} \right] D^* (Ty, Tz) + \beta D^* (x, y, z)
\]

for all \(x, y, z \in X\), where \(\alpha \geq 0, \beta \geq 0\) with \(\alpha + \beta < 1\). Then \(T\) has unique fixed point in \(X\).
Proof: Let $x_0 \in X$.

Define $x_{n+1} = Tx_n$, $n = 0, 1, 2, 3, \ldots$
If $x_{n+1} = x_n$ for some $n$, then $x_n$ is a fixed point of $T$.
Assume that $x_{n+1} \neq x_n$ for all $n$.

Then

$$D^\ast(x_{n+1}, x_{n+1}, x_{n+1}) = D^\ast(Tx_n, Tx_n, Tx_n) \leq \alpha \left[ \frac{1 + D^\ast(x_{n-1}, x_n, x_n)}{1 + D^\ast(x_{n-1}, x_n, x_n)} \right] D^\ast(x_n, x_{n+1}, x_{n+1}) + \beta D^\ast(x_{n-1}, x_n, x_n).$$

Thus

$$D^\ast(x_n, x_{n+1}, x_{n+1}) \leq \frac{\beta}{1-\alpha} D^\ast(x_{n-1}, x_n, x_n).$$

Now from $\left(D^\ast\right)_1$, we have

$$D^\ast(x_{n+1}, x_n, x_n) \leq \lambda D^\ast(x_n, x_{n-1}, x_{n-1}), \text{ where } \lambda = \frac{\beta}{1-\alpha} < 1.$$

Continuing this way, we get

$$D^\ast(x_{n+1}, x_n, x_n) \leq \lambda^n D^\ast(x_1, x_0, x_0).$$

Now for $m > n$, using $(D^\ast)_2$ repeatedly, we get

$$D^\ast(x_m, x_n, x_n) \leq D^\ast(x_{n+1}, x_n, x_n) + D^\ast(x_{n+2}, x_n, x_n) + \ldots + D^\ast(x_{m}, x_{m-1}, x_{m-1}) \leq (\lambda^n + \lambda^{n+1} + \ldots + \lambda^{m-1}) D^\ast(x_1, x_0, x_0) \leq \frac{\lambda^n}{1-\lambda} D^\ast(x_1, x_0, x_0) \to 0 \text{ as } n \to \infty, m \to \infty.$$

Hence $\{x_n\}$ is Cauchy. Since $(X, D^\ast)$ is a complete dislocated quasi $D^\ast$-metric space, there exists $u \in X$ such that $\{x_n\}$ converges to $u$.
Since $T$ is continuous, we have

$$Tu = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_{n+1} = u.$$  

Thus $u$ is a fixed point of $T$.

Uniqueness: Let $x$ be a fixed point of $T$.
Then

$$D^\ast(x, x, x) = D^\ast(Tx, Tx, Tx)$$
\[
\leq \alpha \left[ \frac{1 + D^*(x,x,x)}{1 + D^*(x,x,x)} \right] D^*(x,x,x) + \beta D^*(x,x,x)
= (\alpha + \beta) D^*(x,x,x)
\]

Since \(0 \leq \alpha + \beta < 1\), we have \(D^*(x,x,x) = 0\).

Thus if \(x\) is a fixed point of \(T\), then \(D^*(x,x,x) = 0\).

Let \(x\) and \(y\) be fixed points of \(T\).
Then \(D^*(x,x,x) = 0 = D^*(y,y,y).\) Now
\[
D^*(x,y,y) = D^*(T_x,T_y,T_y)
\leq \alpha \left[ \frac{1 + D^*(x,y,y)}{1 + D^*(y,y,y)} \right] D^*(y,y,y) + \beta D^*(x,y,y)
= \beta D^*(x,y,y).
\]

Since \(0 \leq \beta < 1\), we have \(D^*(x,y,y) = 0.\) Hence \(x = y\).

Thus the fixed point of \(T\) is unique.

Now we give a coincidence point theorem for four mappings in dislocated \(D^*\)-metric spaces.

**Theorem 2.2:** Let \((X, D^*)\) be a complete dislocated \(D^*\)-metric space. Let \(A, B, S, T : X 
\rightarrow X\) be \(D^*\)-continuous mapping satisfying

\begin{align*}
& (2.2.1) \ AS = SA, \ BT = TB, \\
& (2.2.2) \ A(X) \subseteq T(X), \ B(X) \subseteq S(X), \\
& (2.2.3) \ D^*(Ax, By, z) \leq h \max \{D^*(Sx, Ty, z), D^*(Sx, Ax, z), D^*(Ty, By, z)\}
\end{align*}

for all \(x, y, z \in X\), \(0 \leq h < 1\).

Then (i) \(A\) and \(S\) or \(B\) and \(T\) have a coincidence point in \(X\) or

(ii) The pairs \((A, S)\) and \((B, T)\) have a common coincidence point.

**Proof:** Let \(x_0 \in X\).

Define \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that
\[
y_{2n} = Ax_{2n} = Tx_{2n+1}, \ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}, \ n = 0, 1, 2, 3, \ldots.
\]
Suppose \(y_{2n} = y_{2n+1}\) for some \(n\).
Then \(Tx_{2n+1} = Bx_{2n+1}\). Hence \(x_{2n+1}\) is a coincidence point of \(T\) and \(B\).
Suppose \(y_{2n+1} = y_{2n+2}\) for some \(n\).
Then \(Sx_{2n+2} = Ax_{2n+2}\). Hence \(x_{2n+2}\) is a coincidence point of \(S\) and \(A\).
Assume that \(y_{n} \neq y_{n+1}\) for all \(n\).
Denote \(d_n = D^*(y_n, y_{n+1}, y_{n+1})\)
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\[ d_{2n} = D^*(y_{2n}, y_{2n+1}, y_{2n+1}) = D^*(y_{2n}, y_{2n+1}, y_{2n}) \text{ from } (D^*_3) \text{ and } (D^*_4) \]

\[ = D^*(Ax_{2n}, Bx_{2n+1}, y_{2n}) \]
\[ \leq h \max \{ D^*(y_{2n-1}, y_{2n}, y_{2n}), D^*(y_{2n-1}, y_{2n}, y_{2n}), D^*(y_{2n}, y_{2n+1}, y_{2n}) \} \]
\[ \leq h \max \{ d_{2n-1}, d_{2n-1}, d_{2n} \} \text{ from } (D^*_3) \text{ and } (D^*_4) \]

Thus \( d_{2n} \leq h d_{2n-1}. \)

\[ d_{2n+1} = D^*(y_{2n+2}, y_{2n+2}, y_{2n+2}) = D^*(y_{2n+2}, y_{2n+1}, y_{2n+1}) \text{ from } (D^*_3) \]
\[ = D^*(Ax_{2n+2}, Bx_{2n+1}, y_{2n+1}) \]
\[ \leq h \max \{ D^*(y_{2n+1}, y_{2n+1}, y_{2n+1}), D^*(y_{2n+1}, y_{2n+2}, y_{2n+1}), D^*(y_{2n}, y_{2n+1}, y_{2n+1}) \} \]
\[ \leq h \max \{ d_{2n}, d_{2n+1}, d_{2n} \} \text{ from } (D^*_3) \text{ and } (D^*_4) \]

Thus \( d_{2n+1} \leq h d_{2n}. \)

Hence \( d_n \leq h d_{n-1} \) for \( n = 1, 2, 3, \ldots \)

Hence \( d_n \leq h^n d_0 = h^n D^*(y_0, y_1, y_1) \)

Now for \( m > n \) and using \( (D^*_2), (D^*_3), (D^*_4) \) repeatedly we have

\[ D^*(y_{2n}, y_{2n}, y_m) \leq D^*(y_{2n}, y_{2n}, y_{n+1}) + D^*(y_{n+1}, y_{n+1}, y_{n+2}) + \ldots + D^*(y_{m-1}, y_{m-1}, y_m) \]
\[ = d_n + d_{n+1} + \ldots + d_{m-1} \]
\[ \leq (h^n + h^{n+1} + \ldots + h^{m-1}) D^*(y_0, y_1, y_1) \]
\[ \leq \frac{h^n}{1-h} D^*(y_0, y_1, y_1) \]
\[ \to 0 \text{ as } n \to \infty, m \to \infty. \]

Thus \( \{y_n\} \) is a Cauchy sequence in the complete dislocated \( D^* \)-metric space \( X \).

Hence there exists \( u \in X \) such that \( y_n \to u. \)

Clearly the sub sequences \( \{Ax_{2n}\} \to u, \{Bx_{2n+1}\} \to u, \{Tx_{2n+1}\} \to u \) and \( \{Sx_{2n+2}\} \to u. \)

Since \( AS = SA \) and \( A \) and \( S \) are continuous, we have

\[ Au = \lim_{n \to \infty} ASx_{2n} = \lim_{n \to \infty} SAx_{2n} = Su. \]

Since \( BT = TB \) and \( B \) and \( T \) are continuous, we have

\[ Bu = \lim_{n \to \infty} BTx_{2n+1} = \lim_{n \to \infty} TxBx_{2n+1} = Tu. \]

Thus \( u \) is a common coincidence point of the pairs \( (A, S) \) and \( (B, T). \)

**Theorem 2.3:** Let \( (X, D^*) \) be a complete dislocated \( D^* \)-metric space. Let \( A, B : X \to X \) be \( D^* \)-continuous mapping satisfying

\[ D^*(Ax, By, z) \leq h \max \{ D^*(x, y, z), D^*(x, Ax, z), D^*(y, By, z) \} \]

for all \( x, y, z \in X \) and \( 0 \leq h < 1. \)

Then either \( A \) or \( B \) a fixed point or \( A \) and \( B \) have a unique common fixed point.

**Proof:** Putting \( S = T = I \) (Identity map) in Theorem 2.2., we have either \( A \) or \( B \) has a
fixed point or A and B have a common fixed point.

Suppose u and v are two common fixed points of A and B.
\[ D^*(u,u,v) = D^*(Au,Bu,v) \]
\[ \leq h \max \{D^*(u,u,v), D^*(u,u,v), D^*(u,u,v)\} \]
\[ = h D^*(u,u,v). \]

Since \( 0 \leq h < 1 \), we have that \( D^*(u,u,v) = 0 \).
Hence \( u = v \).
Thus A and B have a unique common fixed point.

References