Global Journal of Mathematical Sciences: Theory and Practical.
ISSN 0974-3200 Volume 3, Number 4 (2011), pp. 347-354

http://www.irphouse.com

A Common Fixed Point Theorem in Cone Metric Spaces under S-Type Control Function

K.P.R. Sastry[1], K.K.M. Sarma[2], Ch. Srinivasarao[3], K. Sujatha[4] and G. Praveena[5]

[1]*8-28-8/1, Tamil Street, Chinna Waltair, Visakhapatnam-530 017, India*
E-mail: kprsastry@hotmail.com
[2]*Department of Mathematics, Andhra University, Visakhapatnam-530 003, India.*
E-mail: sarmakmkandala@yahoo.in
[3]*Department of Mathematics, Mrs. A.V.N. College, Visakhapatnam-530 001, India.*
E-mail: drcsr41@yahoo.com
[4]*Department of Mathematics, St. Joseph's College for Women, Visakhapatnam-530 004, India*
E-mail: kambhampati.sujatha@yahoo.com
[5]*Department of Mathematics, St. Paul College, Visakhapatnam-530 001, India*
E-mail: praveenagorapalli29@gmail.com

Abstract

In this paper we prove the existence of coincidence points and common fixed points for large class of a almost contractions in cone metric spaces and obtain results of Berinde as a corollary in S -cone metric spaces.

AMS subject classification 2000: 47H10, 54H25

Keywords: S-cone metric space, weakly compatible mappings, contractive type mappings, iterative method, S-type control function.

Introduction

In this section, we introduce the familiar notions of cone metric spaces, and state the results of Berinde, which we need in the next section.

Definition 1.1 (S. Rezapour [7]): Let E be a real Banach space and P a subset of E. P is called a cone if

(i) P is closed, non empty and $P \neq \{0\}$;
(ii) $ax + by \in P \; \forall \, x, y \in P$ and non negative real numbers a, b;

(iii) $P \cap (-P) = \{0\}$.

Note also that the relations $int\, P + int\, P \subseteq int\, P$ and $\lambda\, int\, P \subseteq int\, P$ $(\lambda > 0)$ hold. For a given cone $P \subseteq E$, we can define on E a partial ordering $\leq$ with respect to P by putting $x \leq y$ if and only if $y - x \in P$. Further, $x < y$ stands for $x \leq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in int\, P$, where $int\, P$ denotes the interior of P.

Definition 1.2 (L.G. Huang [5]): Let X be a nonempty set. A mapping $d: X \times X \to E$ satisfying

(i) $0 \leq d(x, y)\ \forall\, x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)\ \forall x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)\ \forall x, y, z \in X$,

is called a cone metric on X, while (X,d) is called a cone metric space.

Example 1.3: Let $(X, d_1), (X, d_2)$ be two metric spaces (that is, d_1, d_2 are two metrics on the same underlying space X). Let $E = R^2$ be the Euclidean plane and

$P = \{(x, y) \in R^2 : x, y \geq 0\}$. Then P is a cone in E. Define $d: X \times X \to P$ by

$d(x, y) = (d_1(x, y), d_2(x, y))\ \forall\, x, y \in X$; Then (X, d) is a cone metric space.

Note: If $\alpha, \beta > 0$ and if we define $d(x, y) = (\alpha d_1(x, y), \beta d_2(x, y))\ \forall\, x, y \in X$, then d is also cone metric on X.

Definition 1.4 (L.G. Huang [5]): Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n\geq1}$ a sequence in X. Then

(i) $\{x_n\}_{n\geq1}$ converges to x whenever for every $\varepsilon \in E$ with $0 \ll \varepsilon$, there is a natural number N such that $d((x_n, x) \ll \varepsilon$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, as in the usual case.
(ii) $\{x_n\}_{n\geq1}$ is a Cauchy sequence whenever for every $\varepsilon \in E$ with $0 \ll \varepsilon$ there is a natural number N such that $d(x_{n+p}, x_n) \ll \varepsilon$ for all $n \geq N$ and all p;
(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Definition 1.5 (M. Abbas [1]): Let S and T be self maps of a nonempty set X. If there exists $x \in X$ such that $Sx = Tx$ then x is called a coincidence point of S and T, while $y = Sx = Tx$ is called a point of coincidence of S and T.

If $Sx = Tx = x$, then x is a common fixed point of S and T.

Definition 1.6 (G. Jungck [6]): Let S and T be self maps of a nonempty set X. The pair (S, T) of mappings is said to be weakly compatible if they commute at their coincidence points.

The next proposition (M. Abbas and G. Jungck [1], Proposition 1.4) will be needed in our main result.

Proposition 1.7: Let *S* and *T* be weakly compatible self maps of a nonempty set *X*.

If S and T have a unique point of coincidence $y = Sx = Tx$, then *y* is the unique common fixed point of *S* and *T*.

Berinde [3] proved the following theorem for an almost contractive self map on a complete metric space

Theorem 1.8 (V. Berinde [4], Theorem 1): Let (*X, d)* be a complete metric space and $T : X \to X$ an almost contraction, that is a mapping for which there exist a constant $\delta \in (0,1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta \,.\, d(x, y) + Ld(y, Tx), for\ all\ x, y \in X \ldots \quad (1.8.1)$$

Then

1. $F(T) = \{x \in X : Tx = x\} \neq \emptyset$;
2. For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = Tx_n$ converges to some $x^* \in$ *F(T);*
3. The following estimate holds $d(x_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta} d(x_n, x_{n-1})$, $n = 0,1,2 \ldots, i = 1,2 \ldots$.

This theorem concludes that *T* has a fixed point. However, the fixed point need not be unique in view of the following example.

Example 1.9: Define $T: \{0,1\} \to \{0,1\}$ by $T(x) = \begin{cases} 0\ if\ x = 0 \\ 1\ if\ x = 1 \end{cases}$

Then T satisfies (1.8.1) and T has two fixed points.

Berinde [4] extended this result as a coincidence theorem to two self maps on a cone metric space(X, d) as follows

Theorem 1.10 (V. Berinde [4], Theorem 2): Let (*X, d*) be a cone metric space and let $T, S : X \to X$ be two mappings for which there exist a constant $\delta \in (0,1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta\, d(Sx, Sy) + Ld(Sy, Tx), for\ all\ x, y \in X \quad (1.10.1)$$

If the range of *S* contains the range of *T* and *S(X*) is a complete subspace of *X*, then *T* and *S* have a coincidence point in *X*. Moreover, for any $x_0 \in X$, the iteration $\{Sx_n\}$ defined by $Sx_{n+1} = Tx_n$ converges to some coincidence point x^* of *T* and *S*.

The coincidence point obtained from theorem (1.10) need not be unique in view of example (1.9) (by taking $S = T$).

In order to obtain a common fixed point theorem from the above coincidence point theorem Berinde [4] imposed an additional contractive condition which makes the coincidence point unique and hence becomes a common fixed point.

Theorem 1.11 (V. Berinde [4], Theorem 3): Let (*X, d*) be a cone metric space and let $T, S : X \to X$ be two mappings satisfying (1.10.1) for which there exist a constant

$\theta \in (0,1)$ and some $L_1 \geq 0$ such that

$$d(Tx,Ty) \leq \theta d(Sx,Sy) + L_1\, d(Sx,Tx), \text{ for all } x,y \in X\,, \tag{1.11.1}$$

If the range of S contains the range of T and *S(X)* is a complete subspace of *X*, then *T* and *S* have a unique coincidence point in *X*. Moreover, if *T* and *S* are weakly compatible, then *T* and *S* have a unique common fixed point in *X*. In both cases, for any $x_0 \in X$, the iteration $\{Sx_n\}$ defined by $Sx_{n+1} = Tx_n$ converges to the unique common fixed point (coincidence point) x^* of *S* and *T*.

Babu et.al [3] unified (1.10.1) and (1.11.1) in the metric space context for a single map and obtained the following theorem.

Theorem 1.12 (G. V. R. Babu [2], Theorem 2.3): Let *(X, d)* be a complete metric space and let $T : X \to X$be a map satisfying the condition

$$d(Tx,Ty) \leq \delta\, d(x,y) + L\, min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

Then T has a unique common fixed point.

In the cone metric space context of the above theorem, Berinde [4] has proved the following Theorem 1.13.

Theorem 1.13 (V. Berinde [4], Theorem 4): Let (*X*, *d*) be a cone metric space and let $T, S : X \to X$ be two mappings for which there exist a constant $\delta \in (0,1)$ and some $L \geq 0$ such that

$$d(Tx,Ty) \leq \delta d(Sx,Sy) + L\, min\{d(Sx,Tx), d(Sy,Ty), d(Sx,Ty), d(Sy,Tx)\} \tag{1.13.1}$$

for all $x,y \in X$. If the range of S contains the range of T and *S*(*X*) is a complete subspace of *X*, then *T* and *S* have a unique coincidence point in *X*. Moreover, if *T* and S are weakly compatible, then *T* and *S* have a unique common fixed point in *X*. In both cases, for any $x_0 \in X$, the iteration $\{Sx_n\}$ defined by $Sx_{n+1} = Tx_n$ converges to the unique common fixed point (coincidence point) x^* of S and T.

Note: In Theorem 1.13 the right hand side of the equation (1.13.1) may not be meaningful, since $min\{d(Sx,Tx), d(Sy,Ty), d(Sx,Ty), d(Sy,Tx)\}$ may not exist in *P* (cone of *E*).

In order to overcome this difficulty, in the next section we introduce the concept of S-type control function, and obtain a satisfactory account of the above theorem.

Main results

In this section we introduce the concept of S-type control function and obtain a satisfactory account of theorem (1.13*)*

Definition 2.1: Let *E* be a real Banach space and *P* a cone in *E*. Suppose $\varphi: P^4 \to P$ is a continuous function which satisfies the condition.

(S) : $\varphi(t_1, t_2, t_3, t_4) = 0$ if any one of t_1, t_2, t_3, t_4 is zero.

Then φ is called a S– type control function.

Example 2.2: Let E be a Banach space, P be a cone in E. Let t_1, t_2, t_3, t_4 be bounded linear functionals on E. Define φ: $P^4 \to P$ by

$$\varphi(t_1, t_2, t_3, t_4) = |f_1(t_1).f_2(t_2).f_3(t_3).f_4(t_4)|(t_1 + t_2 + t_3 + t_4)$$

Then φ is continuous and satisfies $\varphi(t_1, t_2, t_3, t_4) = 0$ if any one of t_1, t_2, t_3, t_4 is zero. Thus φ is a S - type control function.

Definition 2.3: Let (X, d) be a cone metric space with normal cone P and normal constant K. Suppose $S = \{d(x, y): x, y \in X\}$ is a totally ordered subset of P. Then (X, d) is called a S –cone metric space.

Example 2.4: Let $E = R^2$, $P = \{(x, y) \in E{:}x \geq 0, y \geq 0\}$ and $X = [0,1]$.

Define $d: X \times X \to P$ by $d(x, y) = (|x - y|, \frac{1}{2}|x - y|)$.

Then *(X, d)* is a S –cone metric space.

We observe that every metric space is a S - cone metric space.

Example 2.5: Let (X, d_1) be a metric space and $\alpha > 0$. Let E and P be as in Example 2.4. Define $d: X \times X \to P$ by $d(x, y) = (d_1(x, y), \alpha d_1(x, y))\ \forall\, x, y \in X$.

Then (X, d) is a S –cone metric space.

Example 2.6: Let (X,d) be a S – cone metric space and let S,T be self maps on X. Define $\varphi(x, y) = min\{d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\}\ \forall x, y \in X$, then φ is a S - type control function.

Theorem 2.7: Let *(X, d)* be a cone metric space and let $T, S : X \to X$ be two mappings for which there exist a constant $\delta \in (0, 1)$ and a S – type control function φ such that

$$d(Tx, Ty) \leq \delta d(Sx, Sy) + \varphi\{d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\} \quad (2.7.1)$$

$\forall\, x, y \in X$. If the range of S contains the range of T and $S(X)$ is a complete subspace of X, then T and S have a unique coincidence point in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X. In both cases, for any $x_0 \in X$, the iteration $\{Sx_n\}$ defined by $Sx_{n+1} = Tx_n$ converges to the unique common fixed point (coincidence point) x^* of S and T.

Proof: Let x_0 be an arbitrary point in X. Since $T(X) \subset S(X)$, we can choose a point x_1 in X such that $Tx_0 = Sx_1$. Continuing in this way, for a x_n in X, we can find

$$x_{n+1} \in X \text{ such that } Sx_{n+1} = Tx_n,\ n = 0,1,2 \quad (2.7.2)$$

If $x = x_{n-1}$, $y = x_n$ are two successive terms of the sequence defined by(2.7.2), then by (2.7.1) we have

$d(Tx_{n-1}, Tx_n) \leq \delta\, d(Sx_{n-1}, Sx_n)$
$+\varphi\{d(Sx_{n-1}, Tx_{n-1}), d(Sx_n, Tx_n), d(Sx_{n-1}, Tx_n), d(Sx_n, Tx_{n-1})\}$
$\Rightarrow d(Tx_{n-1}, Tx_n) \leq \delta\, d(Tx_{n-2}, Tx_{n-1})$
$+\varphi\{d(Tx_{n-2}, Tx_{n-1}), d(Tx_{n-1}, Tx_n), d(Tx_{n-2}, Tx_n), d(Tx_{n-1}, Tx_{n-1})\}$

In view of (2.7.2). since $d(Tx_{n-1}, Tx_{n-1}) = 0$, the above equation reduces to
$d(Tx_{n-1}, Tx_n) \leq \delta\, d(Tx_{n-2}, Tx_{n-1})$
$d(Tx_{n-1}, Tx_n) \leq \delta^2 d(Tx_{n-3}, Tx_{n-2})$

Hence, in general, we have
$$d(Tx_{n-1}, Tx_n) \leq \delta^{n-1} d(Tx_0, Tx_1) \tag{2.7.3}$$

Now for p ≥ 1, we get
$d(Tx_{n+p}, Tx_n) \leq d(Tx_{n+p}, Tx_{n+p-1}) + d(Tx_{n+p-1}, Tx_{n+p-2}) + \cdots$
$+d(Tx_{n+1}, Tx_n).$
$\leq \delta^{n+p-1} d(Tx_0, Tx_1) + \delta^{n+p-2}\, d(Tx_0, Tx_1) + \cdots + \delta^n d(Tx_0, Tx_1).$
$= \delta^n(\delta^{p-1} + \delta^{p-2} + \cdots + 1)\, d(Tx_0, Tx_1)$
$= \frac{\delta^n(1-\delta^p)}{1-\delta}\, d(Tx_0, Tx_1)$
$\leq \frac{\delta^n}{1-\delta}\, d(Tx_0, Tx_1)$

Let now $0 \ll \varepsilon$ be given. Choose $\lambda > 0$ such that $\varepsilon + N_\lambda(0) \subset int\ P$, where $N_\lambda(0) = \{y \in E: \|y\| < \lambda\}$. Also choose a natural number N_1 such that
$\frac{\delta^n}{1-\delta}\, d(Tx_0, Tx_1) \in N_\lambda(0)\ \forall\, n \geq N_1$

Then $\frac{\delta^n}{1-\delta}\, d(Tx_0, Tx_1) \ll \varepsilon\ \forall\, n \geq N_1$
And hence $d(Tx_{n+p}, Tx_n) \leq \frac{\delta^n}{1-\delta}\, d(Tx_0, Tx_1) \ll \varepsilon, \forall\, n \geq N_1$
which shows that $\{Tx_n\}$ is a Cauchy sequence and hence $\{Sx_n\}$ is also Cauchy.

Since *S(X)* is complete, there exists a x^*in $S(X)$ such that
$$lim_{n\to\infty} T\, x_n = lim_{n\to\infty} S\, x_n = x^* \ldots. \tag{2.7.4}$$

We can find $p \in X$ such that $Sp = x^*$ (since $x^* \in S(X)$)
Now we show that $\{Tx_n\} \to Tp$
We have, by (2.7.1)
$d(Tx_n, Tp) \leq \delta d(Sx_n, Sp) + \varphi\{d(Sx_n, Tx_n), d(Sp, Tp), d(Sx_n, Tp), d(Sp, Tx_n)\}$
Letting $n \to \infty$, we get that

$$lim_{n\to\infty} d(Tx_n, Tp) \le lim_{n\to\infty}(\delta d(Sx_n, Sp) + \varphi\{d(Sx_n, Tx_n), d(Sp, Tp), d(Sx_n, Tp), d(Sp, Tx_n)\})$$

$$= lim_{n\to\infty}\delta d(Sx_n, Sp) + \varphi\{lim_{n\to\infty}d(Sx_n, Tx_n), d(Sp, Tp), lim_{n\to\infty}d(Sx_n, Tp), lim_{n\to\infty}d(Sp, Tx_n)\}$$
$$= \delta d(x^*, x^*) + \varphi\{d(x^*, x^*), d(x^*, Tp), d(x^*, Tp), d(x^*, x^*)\}$$
$$= 0.$$

So that $\lim_{n\to\infty} d(Tx_n, Tp) \le 0$

Hence $lim_{n\to\infty} d(Tx_n, Tp) = 0$

Thus $Tx_n \to Tp$ … (2.7.5)

By (2.7.4) and (2.7.5) follows that $Tp = Sp = x^*$

i.e. p is a coincidence point of *T* and *S*. (or x^* is a point of coincidence of *T* and *S*).

Now we prove that x^* (point of coincidence of *T* and *S*) is unique.

Let $Tx = Sx = y^*$ and $Sp = Tp = x^*$ be two points of coincidence of T and *S*. Then, we show that $x^* = y^*$

We have, from

$$d(Tx, Tp) \le \delta d(Sx, Sp) + \varphi\{d(Sx, Tx), d(Sp, Tp), d(Sx, Tp), d(Sp, Tx)\}$$
$$= \delta d(Tx, Sp) + \varphi\{0, 0, d(Sx, Tp), d(Sp, Tx)\}$$
$$\Rightarrow d(Tx, Tp) \le \delta d(Tx, Sp) + 0 = \delta d(Tx, Tp)$$
$$\Rightarrow d(Tx, Tp) = 0$$
$$\Rightarrow Tx = Tp$$
$$\Rightarrow x^* = y^*.$$

Thus x^* is the unique point of coincidence of *T* and *S*.

Now suppose *T* and *S* are weakly compatible.

Then, by Proposition 1.7, x^* is the unique point of coincidence of *T* and *S*.

The following example supports Theorem 2.7

Example 2.8: Let (*X*, *d*) be a complete cone metric space and $x_0 \in X$ be fixed. Let $T, S: X \to X$ be defined by $Tx = x_0$ for every x in *X*, and $Sx = x$ for every x in *X*. Then T and S satisfy condition (2.7.1) and also x_0 is the unique common fixed point of *T* and *S*.

Particular case

Theorem 1.13, the main result of Berinde [4] is a particular case of our main result. It follows as a corollary to our main result (Theorem 2.7) by taking the S -type control function $\varphi: P^4 \to P$ defined by $\varphi(t_1, t_2, t_3, t_4) = \min\{t_1, t_2, t_3, t_4\}$ assuming that the minimum is meaningful.

References

[1] M. Abbas and G. Jungck: Common fixed point results for non commuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl., 341 (2008), 416-420.

[2] G.V.R. Babu, M.L. Sandhya, M.V.R. Kameswari: A note on a fixed point theorem of Berinde on weak contractions, Carpathian J. Math. 24 (2008), 8-12.

[3] V. Berinde: Approximation fixed points of weak contractions using the Picard iteration, Nonlinear Analysis Forum 9 (2004), 43-53.

[4] V. Berinde: Common fixed points of non commuting almost contractions in cone metric spaces, Math. Commun., Vol. 15, no. 1, (2010), 229-241.

[5] L.G. Huang, Z. Xian: Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl., 332 (2007), 1468-1476.

[6] G. Jungck: Common fixed points for non continuous non self maps on non-metric spaces, Far East J. Math. Sci. 4 (1996), 199-215.

[7] S. Rezapour, R. H. Haghi: Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", J. Math. Anal. Appl., 345 (2008), 719-724.