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A Common Fixed Point Theorem for Four Self Maps Controlled by a Generalized Altering Distance Function and a Weak Generalized Altering Distance Function

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Abstract

A new notion of a weak generalized altering distance function is introduced and this notion is used to prove a common fixed point theorem which improves results of K.P.R. Rao et. al. [4] and Van Luong et. al. [3]

Mathematics Subject Classification: 47H10, 54H25

Keywords: Common fixed point, weakly compatible maps, generalized altering distance function, semi compatibility, weak generalized altering distance function.

Introduction

In this paper we introduce the notion of a weak altering distance function and use it to prove a common fixed point result which improves results of K.P.R. Rao et. al. [4] and Van Luong et. al. [3]

Definition 1.1 (Jungck and Rhoades, [2]): Suppose S and T are self maps on a non empty set. S and T are said to be weakly compatible if $STx = TSx$ when ever $Sx = Tx$. That is, they commute at their coincidence point.

Definition 1.2: (Generalized altering distance function): Let Φ denote the set of all functions $\phi : [0,\infty)^5 \to [0,\infty)$ such that

(i) φ is continuous

(ii) φ is monotone increasing in each variable

(iii) $\varphi(t_1, t_2, t_3, t_4, t_5) = 0$ *if and only if* $t_1 = t_2 = t_3 = t_4 = t_5 = 0$

A function $\varphi \in \Phi$ is said to be a generalized altering distance function.

If φ: $[0,\infty)^5 \to R$ satisfies only (ii) and (iii) then φ is called a weak generalized altering distance function.

Definition 1.3 (Cho et. al. [1]): A pair of self maps (S, T) is said to be semi compatible if
$Sy = Ty$ implies $STy = TSy$, and

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$ for some $x \in$ X implies $\lim_{n\to\infty} STx_n = Tx$ holds.

We observe that semi compatibility implies weak compatibility.

K.P.R. Rao et. al. [4] proved a common fixed point theorem for four self maps on a complete metric space using semi compatibility assuming one of the maps to be continuous.

Theorem 1.4 (K.P.R. Rao et. al. [4]): Let (X, d) be a complete metric space and f, g, S and T be self maps on X such that

(i) $$\varphi_1\left(d\left(fx, gy\right)\right) \le \phi_1\left(d\left(Sx,Ty\right), d\left(Sx, fx\right), d\left(Ty, gy\right), \frac{1}{2}\left[d\left(Sx, gy\right) + d\left(Ty, fx\right)\right]\right)$$
$$- \varphi_2\left(d(Sx,Ty), d(Sx, fx), d(Ty, gy), \frac{1}{2}[d(Sx, gy) + d(Ty, fx)]\right)$$

for all x, y $\in$ X, where $\varphi_1, \varphi_2 \in \Phi$ and $\varphi_1(x) = \varphi_1(x, x, x, x) \quad \forall x \in [0,\infty)$

(ii) One of the maps *f, g*, S and T is continuous

(iii) (f, S) and (g, T) are semi compatible pairs

(iv) $f(X) \subset T(X)$, $g(X) \subset S(X)$.

Then *f, g*, S and T have a unique common fixed point in X.

Van Luong and Xuan Thuan [3] proved a common fixed point theorem for two pairs of weakly compatible maps using the notion of generalized altering distance function. In this theorem completeness of the space is replaced by the completeness of the range space of one of the maps and continuity of the maps is dropped. Further they claimed that the continuity of one of the generalized altering distance functions is dropped. However, in the proof, continuity of both the generalized altering distance functions is tacitly used. Further, one need to put $\psi(u,u)=\max\{\varphi_1(u,u,u,2u,u),\varphi_2(u,u,u,u,2u)\}$ to get $a_n \le a_{n-1}$, $n = 2, 3, \ldots$

Theorem 1.5 (Van Luong et. al [3]): Let A, B, S and T be self maps of a metric space (X, d) such that

$$(i)\quad \psi\big(d(Ax,By), \max\{d(Ty,By), d(Sx,Ax), d(Sx,Ty)\big)$$
$$\le \varphi_1(d(Sx,Ty), d(Sx,Ax), d(Ty,By), d(Ty,Ax), d(Sx,By)$$
$$-\varphi_2(d(Sx,Ty), d(Sx,Ax), d(Ty,By), d(Ty,Ax), d(Sx,By) \quad \forall x,y\in X,$$

for some $\varphi_1,\varphi_2\in\Phi$. $\psi:\ [0,\infty)^2\to R$ is a continuous function and

$$\psi(u,u)=\{\varphi_1(u,u,u,u,2u)\}\quad \forall\ u\in[0,\infty)$$

(ii) $AX \subset TX$ and $BX \subset SX$

(iii) (A, S) and (B, T) are weakly compatible

(iv) One of AX, BX, SX or TX is a complete subspace of X.

Then A, B, S and T have a unique common fixed point. Moreover it is also the unique common fixed point of A and S and of B and T.

We overcome these difficulties in Section 2 and obtain our main result.

Main Results

In this section, we prove our main result from which we get the results of [3] and [4] as Corollaries. We start with

Theorem 2.1: Suppose $\varphi_1\in\Phi$ and φ_2 is a weak generalized altering distance function.

Define $f:\ [0,\infty)\to R$ by

$$f(u)=\max\{\varphi_1(u,u,u,0,2u),\varphi_1(u,u,u,u,2u,0),\varphi_1(u,u,u,u,u)\}$$

so that f is increasing. Let A, B, S, T be self maps of a metric space (X, d) such that

$$(i) f(\min\{(d(Ax,By), \max\{d(Ty,By), d(Sx,Ax), d(Sx,Ty)\}\})$$
$$\le \varphi_1(d(Sx,Ty), d(Sx,Ax), d(Ty,By), d(Ty,Ax), d(Sx,By)$$
$$-\varphi_2\ (d(Sx,Ty), d(Sx,Ax), d(Ty,By), d(Ty,Ax), d(Sx,By)\quad \forall\ x,y\in X$$

(ii) $AX \subset TX$ and $BX \subset SX$

(iii) (A, S) and (B, T) are weakly compatible

(iv) One of $AX, BX, SX\ or\ TX$ is a complete subspace of X.

Then $A, B, S\ and\ T$ have a unique common fixed point. Moreover it is also the unique common fixed point of A and S and of B and T.

Proof: Let $x_0 \in X$. Using (ii), we construct the sequences $\{x_n\}$ and $\{y_n\}$ in X, such that, $y_{2n} = Ax_{2n} = Tx_{2n+1}$; $y_{2n+1} = Bx_{2n+1} = Sx_{2n+1}$, $n = 0,1,2..$

Let $a_n = d(y_n, y_{n+1})$, $n = 0, 1, 2, ...$ Put $x = x_{2n}, y = x_{2n+1}$ in (i). We have

$$f(\min\{d(y_{2n}, y_{2n+1}), \max\{d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n})\}\})$$
$$\leq \varphi_1(d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n}), d(y_{2n-1}, y_{2n+1}))$$
$$-\varphi_2(d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n}), d(y_{2n-1}, y_{2n+1}))$$
$$\Rightarrow f(\min\{a_{2n}, \max\{a_{2n}, a_{2n-1}, a_{2n+1}\}\}) \leq \varphi_1(a_{2n-1}, a_{2n-1}, a_{2n}, 0, a_{2n+1} + a_{2n})$$
$$-\varphi_2(a_{2n-1}, a_{2n-1}, a_{2n}, 0, d(y_{2n-1}, y_{2n+1}))$$

If $a_{2n-1} < a_{2n}$ then

$$f(a_{2n}) \leq \varphi_1(a_{2n}, a_{2n}, a_{2n}, 0, 2a_{2n}) - \varphi_2(a_{2n-1}, a_{2n-1}, a_{2n}, 0, d(y_{2n-1}, y_{2n+1}))$$
$\leq f(a_{2n})$, a contradiction.

$\therefore a_{2n} \leq a_{2n-1}$

Similarly by putting $x = x_{2n+2}\ \ and\ \ y = x_{2n+1}$ in (i), we get $a_{2n+1} \leq a_{2n}$.

Thus $a_n \leq a_{n-1}$; $n = 1, 2 ...$ Hence $\{a_n\}$ converges to some $a \in [0, \infty)$ so that $a \leq a_n$

$$Now, d(y_{2n-1}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}) = a_{2n-1} + a_{2n} \rightarrow (2.1.1)$$

Let $0 \leq \alpha \leq d(y_{2n-1}, y_{2n+1}) \quad \forall n \rightarrow (2.1.2)$

From (2.1.1) and (2.1.2) we get $\alpha \leq 2a$. Now, since φ_2 is increasing

$$-\varphi_2(a_{2n-1}, a_{2n-1}, a_{2n}, 0, d(y_{2n-1}, y_{2n+1})) \leq -\varphi_2(a, a, a, 0, \alpha) \quad \rightarrow (2.1.3)$$

Similarly $-\varphi_2(a_{2n}, a_{2n+1}, a_{2n}, d(y_{2n-1}, y_{2n+1}), 0) \leq -\varphi_2(a, a, a, 0, \alpha) \quad \rightarrow (2.1.4)$

Put $x = x_{2n}\ \ and\ \ y = x_{2n+1}$ in (i), Then

$$f(\min\{a_{2n}, \max\{a_{2n}, a_{2n-1}, a_{2n-1}\}\}) \quad \leq \varphi_1(a_{2n-1}, a_{2n-1}, a_{2n}, 0, a_{2n-1} + a_{2n})$$

$$-\varphi_2(a_{2n-1}, a_{2n-1}, a_{2n}, 0, d(y_{2n-1}, y_{2n+1}))$$
$$\Rightarrow f(a_{2n}) \leq f(a_{2n-1}) - \varphi_2(a, a, a, 0, \alpha) \text{ (by (2.1.3))}$$

Hence $\varphi_2(a, a, a, 0, \alpha) \leq f(a_{2n-1}) - f(a_{2n}) \quad \rightarrow (2.1.5)$

Put $x = x_{2n+2}\ \ and\ \ y = x_{2n+1}$ in (i). Then

$f(\min\{a_{2n+1}, \max\{a_{2n+1}, a_{2n}, a_{2n}\}\}) \quad \leq \phi_1(a_{2n}, a_{2n+1}, a_{2n}, a_{2n} + a_{2n+1}, 0)$

$-\varphi_2(a_{2n}, a_{2n+1}, a_{2n}, d(y_{2n}, y_{2n+2}), 0)$

$\Rightarrow f(a_{2n+1}) \quad \leq f(a_{2n}) - \varphi_2(a, a, a, \alpha, 0)$ (by (2.1.4))

Hence $\varphi_2(a, a, a, \alpha, 0) \leq f(a_{2n}) - f(a_{2n+1}) \quad \rightarrow (2.1.6)$

From (2.1.5) and (2.1.6), $\varphi_2(a, a, a, 0, 0) \leq f(a_{2n}) - f(a_{2n+1})$

Hence $n\varphi_2(a, a, a, 0, 0) \leq f(a_1) - f(a_{n+1})$

$\leq f(a_1)$

so that $\varphi_2(a, a, a, 0, 0) \leq \dfrac{f(a_1)}{n} \rightarrow 0 \quad as \quad n \rightarrow \infty$,

Hence $\varphi_2(a, a, a, \alpha, 0) = 0$, consequently $a = \alpha = 0$; Thus

$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0 \rightarrow (2.1.7)$

Now we prove $\{y_n\}$ is a Cauchy. From (2) it is sufficient to show that the sub sequence $\{y_{2n}\}$ is a Cauchy sequence. Suppose $\{y_n\}$ is not a Cauchy sequence then $\exists \varepsilon > 0$ and sub sequences $\{y_{2n(k)}\}$, $\{y_{2m(k)}\}$ such that $n(k) > m(k) \geq k$ and $d(y_{2m(k)}, y_{2n(k)}) \geq \varepsilon \quad and \quad d(y_{2m(k)}, y_{2n(k)-2}) < \varepsilon \rightarrow (2.1.8)$

By choosing $n(k)$ to be the smallest number exceeding $m(k)$ for which (2.1.8) holds.

$$
\begin{aligned}
\varepsilon &\leq d(y_{2m(k)}, y_{2n(k)}) \\
&\leq d(y_{2m(k)}, y_{2n(k)-2}) + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}) \\
&< \varepsilon + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)})
\end{aligned}
$$

Let $k \rightarrow \infty$, using (2.1.7), we get $\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) = \varepsilon \rightarrow (2.1.9)$

$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d(y_{2m(k)}, y_{2m(k)+1})$

Let $k \rightarrow \infty$, using (2.1.7) and (2.1.9) $\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}) = \varepsilon \rightarrow (2.1.10)$

$|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)-1})| \leq d(y_{2n(k)+1}, y_{2n(k)})$

Let $k \rightarrow \infty$, using (2.1.10) and (2.1.7) $\lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1}) = \varepsilon \rightarrow (2.1.11)$

$|d(y_{2n(k)+1}, y_{2m(k)}) - d(y_{2m(k)}, y_{2n(k)})| \leq d(y_{2n(k)}, y_{2n(k)+1})$

Let $k \rightarrow \infty$, using (2.1.11) and (2.1.7) $\lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)}) = \varepsilon \rightarrow (2.1.12)$

Taking $x = x_{2m(k)} \quad and \quad y = x_{2n(k)+1}$ in (i), we get

$$f(\min\{d(y_{2m(k)},y_{2n(k)+1}),\max\{d(y_{2n(k)},y_{2n(k)+1}),d(y_{2m(k)-1},y_{2m(k)}),d(y_{2m(k)-1},y_{2n(k)})\}\})$$
$$\leq\varphi_1(d(y_{2m(k)-1},y_{2n(k)}),d(y_{2m(k)-1},y_{2m(k)}),d(y_{2n(k)},y_{2n(k)+1}),d(y_{2n(k)},y_{2m(k)}),d(y_{2m(k)-1},y_{2n(k)+1}))$$
$$-\varphi_2(d(y_{2m(k)-1},y_{2n(k)}),d(y_{2m(k)-1},y_{2m(k)}),d(y_{2n(k)},y_{2n(k)+1}),d(y_{2n(k)},y_{2m(k)}),d(y_{2m(k)-1},y_{2n(k)+1}))$$

By (2.1.7), (2.1.9), (2.1.10) and (2.1.11), there exists N such that for $k \geq N$
$d(y_{2m(k)-1},y_{2n(k)}) > \varepsilon/2$; $d(y_{2m(k)-1},y_{2m(k)}) \geq 0$; $d(y_{2n(k)},y_{2n(k)+1}) > 0$;
$d(y_{2n(k)},y_{2m(k)}) > \varepsilon/2$ and $d(y_{2m(k)-1},y_{2n(k)+1}) > \varepsilon/2$

Hence for $k \geq N$, we get
$$f(\min\{d(y_{2m(k)},y_{2n(k)+1}),\max\{d(y_{2n(k)},y_{2n(k)+1}),d(y_{2m(k)-1},y_{2m(k)}),d(y_{2m(k)-1},y_{2n(k)})\}\})$$
$$\leq\varphi_1(d(y_{2m(k)-1},y_{2n(k)}),d(y_{2m(k)-1},y_{2m(k)}),d(y_{2n(k)},y_{2n(k)+1}),d(y_{2n(k)},y_{2m(k)}),d(y_{2m(k)-1},y_{2n(k)+1}))$$
$$-\varphi_2(\varepsilon/2,0,0,\varepsilon/2,\varepsilon/2)$$

Now let $k \to \infty$, using (2.1.7), (2.1.9), (2.1.10), (2.1.11), (2.1.12) we get
$$f(\min\{\varepsilon,\max\{0,0,\varepsilon\}\}) \leq \varphi_1(\varepsilon,0,0,\varepsilon,\varepsilon) - \varphi_2(\varepsilon/2,0,0,\varepsilon/2,\varepsilon/2)$$
$$f(\varepsilon) \leq f(\varepsilon) - \varphi_2(\varepsilon/2,0,0,\varepsilon/2,\varepsilon/2)$$
$< f(\varepsilon)$, a contradiction. $\therefore \{y_n\}$ is a Cauchy sequence.

Suppose TX is a complete subspace of X. Then the sub sequence $y_{2n} = Tx_{2n+1}$ is a Cauchy sequence in TX, hence converges to some $u \in TX$. $\exists\ v \in X,\ \ni Tv = u$

Since $\{y_{2n}\} \to u$, *so* $\{y_n\} \to u$, *hence* $\{y_{2n+1}\} \to u$ by the definition of $\{y_n\}$, we have
$$\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = u$$

Put x = x_{2n} *and* $y = v$ in (i), we have
$$f(\min\{d(u,Bv),\max\{d(u,Bv),d(u,u),d(u,u)\}\})$$
$$\leq\varphi_1(d(u,u),d(u,u),d(u,Bv),d(u,u),d(u,Bv)) - \varphi_2(d(u,u),d(u,u),d(u,Bv),d(u,u),d(u,Bv))$$
$$\Rightarrow \quad f(d(u,Bv)) \leq f(d(u,Bv)) - \varphi_2(0,0,d(u,Bv),0,d(u,Bv))$$
$$\Rightarrow \varphi_2(0,0,d(u,Bv),0,d(u,Bv)) = 0$$
$$\therefore u = Bv$$

Since $BX \subset SX$, so $u = Bv \Rightarrow u \in SX$. Hence $\exists w \in X \ni Sw = u$

Similarly set $x = w$ and $y = x_{2n+1}$ in (i), we get $u = Aw$. Since $AX \subset TX$, so $u = Aw \Rightarrow u \in TX$. Hence $\exists v \in X \ni Tv = u$. $Tv = Bv = Aw = Sw = u$

Since $u = Aw = Sw$ and (A,S) is weak compatible, $ASw = SAw$, so $Au = ASw = SAw = Su$ hence $Au = Su$

Since $u = Tv = Bv$ and (B,T) is weak compatible, $BTv = TBv$, so $Bu =$

$BTv = TBv = Tu$ hence $Bu = Tu$

Put $x = u, y = v$ in (i), we have

$$f(\min\{d(Au,Bv),\max\{d(Tv,Bv),d(Su,Au),d(Su,Tu)\}\})$$
$$\leq \varphi_1(d(Su,Tv),d(Su,Au),d(Tv,Bv),d(Tv,Av),d(Su,Bv))$$
$$-\varphi_2(d(Su,Tv),d(Su,Au),d(Tv,Bv),d(Tv,Av),d(Su,Bv))$$
$$f(\min\{d(Au,Bv),\max\{0,0,d(Au,u)\}\})$$
$$\leq \varphi_1(d(Au,u),0,0,d(Au,u),d(Au,u)) - \varphi_2(d(Au,u),0,0,d(Au,u),d(Au,u))$$
$$f(d(Au,u)) \leq f(d(Au,u)) - \varphi_2(d(Au,u),0,0,d(Au,u),d(Au,u))$$
$$\varphi_2(d(Au,u),0,0,d(Au,u),d(Au,u)) = 0,$$ so $Au = u$.

Similarly by using (i), we show that $Bu = u$ by putting $x = w$, and $y = u$

Thus $Au = Bu = Tu = Su = u$. Hence u is a common fixed point of A, B, S, T

If we assume SX is complete, then the argument is similar.

If AX is complete, $\exists\ u \in AX \subset TX$ such that $y_{2n} \to u$ and if BX is complete then $\exists$ u $\in BX \subset SX$ such that $y_{2n} \to u$. In all the above cases the argument is similar.

Uniqueness of common fixed point

Let z be another common fixed point, put $x = u, y = z$ in (i), we get

$$f(\min\{d(u,z),\max\{0,0,d(u,z)\}\})$$
$$\leq \varphi_1(d(u,z),0,0,d(u,z),d(u,z)) - \varphi_2(d(u,z),0,0,d(u,z),d(u,z))$$
$$\Rightarrow\ f(d(u,z)) \leq f(d(u,z)) - \varphi_2(d(u,z),0,0,d(u,z),d(u,z))$$
$$\varphi_2(d(u,z),0,0,d(u,z),d(u,z)) = 0,$$ so that $u = z$

Let x be fixed point of A and S. Put $u = y$ in (i).

$$f(\min\{d(x,u),\max\{d(x,x),d(x,u),d(u,u)\}\})$$
$$\leq \varphi_1(d(x,u),d(x,x),d(u,u),d(u,x),d(x,u)) - \varphi_2(d(x,u),d(x,x),d(u,u),d(u,x),d(x,u))$$
$$\Rightarrow\ f(d(x,u)) \leq f(d(x,u)) - \varphi_2(d(x,u),0,0,d(u,x),d(u,x))$$
$< f(d(x,u))$, a contradiction if $x \neq u$ so $x = u$.

Hence A and S have a unique common fixed point.

Similarly we prove B, T have a unique common fixed point.

Now we use Theorem 2.1 to prove a common fixed point theorem for six maps.

Theorem 2.2: Suppose $\varphi_1 \in \Phi$ and φ_2 is a weak generalized altering distance function.

Define $f:\ [0,\infty) \to R$ by

$f(u) = \max\{\varphi_1(u,u,u,0,2u),\varphi_1(u,u,u,u,2u,0),\varphi_1(u,u,u,u,u)\}$ so that f is increasing.

Let A, B, S, T, P, Q be self maps of a metric space (X, d) such that

$(i)\ f(\min\{\ \ (d(Ax,By),\max\{d(TPy,By),d(SQx,Ax),d(SQx,Ty)\}\})$

$\leq \varphi_1(d(SQx,TPy),d(SQx,Ax),d(TPy,By),d(TPy,Ax),d(SQx,By)$

$-\varphi_2(d(SQx,TPy),d(SQx,Ax),d(TPy,By),d(TPy,Ax),d(SQx,By)\ \ \forall x,y\in X$

(ii) $AX \subset TPX$ and $BX \subset SQX$
(iii) (A,SQ) and (B,TP) are weakly compatible
(iv) $SQ = QS, TP = PT, AQ = QA, BP = PB$
(v) One of $AX, BX, SQX\ or\ TX$ is a complete subspace of X.

Then A,B,S,T,P,Q have a unique common fixed point.

Proof: By using Theorem 2.1, A,B,TP and SQ have a unique common fixed point u, and u is also the unique common fixed point of A and SQ and of B and TP.

Since $Au = SQu = u$ then $QAu = QSQu = Qu$. Thus$AQu = SQQu = Qu$. Qu is a common fixed point of A,SQ. By uniqueness of common fixed point of A and SQ, we get $Qu = u$. Hence $SQu = Su = u$.

Since $Bu = TPu = u$ then $PBu = PTPu = Pu$, thus $BPu = TPPu = Pu$. Hence Pu is a common fixed point of B, TP. By uniqueness of common fixed point of B and TP, we get $Pu = u$. Hence $Tu = TPu = u$.

$Au = Bu = Tu = Su = Qu = Pu = u.$

$i.e.$ A,B,S,T,P,Q have a unique common fixed point.

Note: Clearly, Theorem 1.4 and Theorem 1.5 are corollaries of Theorem 2.1, since semi-compatibility implies weak compatibility.

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