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Estimation of Mortality Rate Deceleration Parameter: Centroid and Polylogarithm Function

[1]E.S. Lakshminarayanan and [2]U. Kumaran

*[1]Professor, School of Mathematics, Madurai Kamaraj University,
Madurai, TamilNadu, India
E-mail: mkueslnmath@yahoo.com
[2]Research scholar, School of Mathematics, Madurai Kamaraj University,
Madurai, Tamil Nadu, India
E-mail: kumara810@gmail.com*

Abstract

Estimation of mortality rate deceleration parameter led us to an equation involving polylogarithm function and that equation has been solved graphically.

Keywords: Logistic Frailty model, Mortality rate deceleration parameter, Centroid, Polylogarithm, Age at which onset of mortality deceleration.

1 INTRODUCTION

The Gompertz hazard function is commonly used to describe age-specific mortality rates in populations and experimental cohorts:

$$\mu(t) = ae^{bt}, \tag{1}$$

where $\mu(t)$ is the instantaneous mortality rate at age t; a is the initial mortality rate, sometimes referred to as the age-independent mortality rate; and b is the age-dependent mortality rate. If $b > 0$, the mortality rate increases indefinitely and exponentially with age. Thus, b is also called the senescence parameter. The Gompertz function has the virtue of simplicity: it has only two parameters, and $ln(\mu(t))$ is a linear function of t (with intercept $ln(a)$ and slope b). Because it is commonly presented in logarithmic form, a and b are also known as the intercept and slope parameters [10]. The Gompertz function provides a fairly good fit for mortality rate data in a wide range of organisms [2]. However, it does not describe deceleration of mortality rates at old ages. The Gompertz hazard function implicitly assumes that all individuals in a cohort or population have the same mortality risk, an assumption that is biologically implausible. Vaupel and his collaborators have developed a class of mortality models that incorporate heterogeneity in individual mortality risk, or frailty [13]. A frailty model based on the Gompertz hazard function is

$$\overline{\mu}_i(t) = z_i a e^{bt},$$

where $\overline{\mu}_i(t)$ is the mortality rate of individual i at age x and z_i is the frailty of individual i. An individual with a frailty of 2, for example, has twice the hazard at any given time as an individual with a frailty of 1. If it is assumed that mean frailty at birth is 1, and that frailty is γ distributed with variance σ^2, then there is a relatively simple expression for the cohort mortality rate at age t:

$$\overline{\mu}(t) = \frac{ae^{bt}}{1 + \frac{a\sigma^2}{b}(e^{bt} - 1)}, \tag{2}$$

where $\overline{\mu}(t)$ is the weighted average of the death rates of the individuals who comprise the population at age t [14]. If $\sigma^2 = 0$, equation (2) reduces to equation (1). If $\sigma^2, b > 0$, then $\mu(t)$ increases exponentially at younger ages but eventually approaches an asymptote equal to $\frac{b}{\sigma^2}$ [10].

Most commonly this mortality deceleration is measured by the life-table aging rate, introduced by Horiuchi and colleagues [5] , but also other methods were used previously. Mortality would decelerate along various trajectories rather than merely plateau [15], and it is better to consider departure from the Gompertz law rather than just convergence of mortality to a plateau level [7].

In [6] we have shown that (the pointwise approximation of Logistic frailty model to Gompertz model) when $\frac{a\sigma^2}{b} < e^{-bt_m}$, the degree of smallness of σ^2 cannot be determined interms of a, b and e^{-bt_m}. In this work, we provide an estimation for σ^2 from areawise approximation of (1) and (2) .

2 Estimation of σ^2

In this paper we are intrested in studying the relation between σ^2 and t^*. In order to find these unknown constants, we are in need of two equations. To obtain the equations , we observe that

1. The area (denoted by A_1) under the curve of Logistic Frailty Mortality function upto t^* is significantly equal to the area (denoted by A_2) under the curve of Gompertz mortality function upto t^*.

2. The centre of mass of A_1 is significantly equal to the centre of mass of A_2.

Then by standard procedure (see Appendix) we get, $\left(\frac{e^{bt^*}(bt^*-1)+1}{b(e^{bt^*}-1)}, \frac{ae^{bt^*}}{4}\right)$ is the centre of mass of A_2. Similiarly for A_1 the centre of mass is

$$\left(t^* - \frac{\int\limits_0^{t^*} ln(1+\frac{a\sigma^2}{b}(e^{bt}-1))dt}{ln(1+\frac{a\sigma^2}{b}(e^{bt^*}-1))}, \frac{b}{2\sigma^2}\left[1-\frac{x}{(1+x)ln(1+x)}\right]\right).$$

If we equate the above two points coordinate wise and after a little algebra we get

$$\int\limits_0^{t^*} ln(1+\frac{a\sigma^2}{b}(e^{bt}-1))dt = \frac{e^{bt^*}-bt^*-1}{b(e^{bt^*}-1)} ln(1+\frac{a\sigma^2}{b}(e^{bt^*}-1)), \quad (3)$$

$$\frac{a\sigma^2}{b}e^{bt^*} = 2[1-\frac{x}{(1+x)ln(1+x)}], \quad (4)$$

where $\quad x = \frac{\frac{a\sigma^2}{b}e^{bt^*}}{1-\frac{a\sigma^2}{b}}$.

Clearly, for a given a and b, $\sigma^2 = 0$ and $t^* = 0$ is the trivial solution of above system.

Since $0 < \frac{x}{(1+x)ln(1+x)}$ and $1 - \frac{x}{(1+x)ln(1+x)} < 1, \forall x > 0$ from (4) we have the estimation $\frac{a\sigma^2}{b}e^{bt^*} < 2$.

In order to improve the estimation rewrite equation (3) as

$$1-\frac{\int_0^{bt^*} ln(1+\frac{a\sigma^2}{b}(e^u-1))du}{ln(1+\frac{a\sigma^2}{b}(e^{bt^*}-1))}-\frac{bt^*}{e^{bt^*}-1}. \tag{5}$$

Considering $\int_0^{bt^*} ln(1+\frac{a\sigma^2}{b}(e^u-1))du$ in (5), integration by parts gives

$$\int_0^{bt^*} ln(1+\frac{a\sigma^2}{b}(e^u-1))du = bt^* ln(1-\frac{a\sigma^2}{b})+\int_{\frac{-\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}}}^{0} \frac{ln(1\quad t)}{t}dt-\int_{\frac{-\frac{a\sigma^2}{b}e^{bt^*}}{1-\frac{a\sigma^2}{b}}}^{0} \frac{ln(1\quad t)}{t}dt. \tag{6}$$

Notice that the right hand side integrals represent polylogarithm functions $Li_2(z)$, where $z=\frac{-\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}}$ and $\frac{-\frac{a\sigma^2}{b}e^{bt^*}}{1-\frac{a\sigma^2}{b}}$. As far as anyone knows, there are exactly eight values of z for which z and $Li_2(z)$ can both be given in closed form [1].
Substituting (6) into (5), we get

$$1+\frac{-bt^* ln(1-\frac{a\sigma^2}{b})-Li_2(-\frac{\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}})+Li_2(-\frac{\frac{a\sigma^2}{b}e^{bt^*}}{1-\frac{a\sigma^2}{b}})}{ln(1+\frac{a\sigma^2}{b}(e^{bt^*}-1))}=\frac{bt^*}{e^{bt^*}-1}. \tag{7}$$

Since $A_1 = A_2$, it follows that

$$ln(1+\frac{a\sigma^2}{b}(e^{bt^*}-1))=\frac{a\sigma^2}{b}(e^{bt^*}-1). \tag{8}$$

In view of (8), equation (7) becomes

$$\frac{a\sigma^2}{b}(e^{bt^*}\quad 1)\quad bt^* ln(1\quad \frac{a\sigma^2}{b})\quad Li_2(\frac{\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}})+Li_2(\frac{\frac{a\sigma^2}{b}e^{bt^*}}{1-\frac{a\sigma^2}{b}})=\frac{a\sigma^2}{b}bt^*$$

or, equivalently

$$\frac{\frac{a\sigma^2}{b}e^{bt^*}+Li_2(-\frac{\frac{a\sigma^2}{b}e^{bt^*}}{1-\frac{a\sigma^2}{b}})-\frac{a\sigma^2}{b}-Li_2(-\frac{\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}})}{bt^*}=\frac{a\sigma^2}{b}+ln(1-\frac{a\sigma^2}{b}).$$

If $\frac{a\sigma^2}{b}$ and $-ln(1-\frac{a\sigma^2}{b})$ are significantly equal (neglecting second and higher order terms), then the above equation reduces to

$$\frac{\frac{a\sigma^2}{b}e^{bt^*} + Li_2(-\frac{\frac{a\sigma^2}{b}e^{bt^*}}{1-\frac{a\sigma^2}{b}}) - \frac{a\sigma^2}{b} - Li_2(-\frac{\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}})}{bt^*} = 0. \tag{9}$$

Since this equation contains polylogarithm functions, we try to obtain a solution graphically.

To find a nontrivial solution of (9) we equate

$$\frac{a\sigma^2}{b}e^{bt^*} + Li_2(\frac{\frac{a\sigma^2}{b}e^{bt^*}}{1-\frac{a\sigma^2}{b}}) = 0 \tag{10}$$

and

$$\frac{a\sigma^2}{b} - Li_2(-\frac{\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}}) = 0. \tag{11}$$

On account of (4), (10) becomes

$$2\left[1-\frac{x}{(1+x)ln(1+x)}\right] + Li_2(-\frac{\frac{a\sigma^2}{b}e^{bt^*}}{1-\frac{a\sigma^2}{b}}) = 0. \tag{12}$$

Solving equations (11) and (12) graphically (see the figures), we get

$$x < 0.0077 \quad and \quad \frac{a\sigma^2}{b} < 0.01. \tag{13}$$

Recalling $x = \frac{\frac{a\sigma^2}{b}e^{bt^*}}{1-\frac{a\sigma^2}{b}}$, it follows that $\frac{\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}} \leq \frac{\frac{a\sigma^2}{b}e^{bt^*}}{1-\frac{a\sigma^2}{b}} \leq 0.0077$ and hence

$$\frac{a\sigma^2}{b} < 0.00764116$$

Now combining with (13), we get

$$\frac{a\sigma^2}{b} \leq \min\{0.01, 0.00764116\} = 0.00764116$$

Putting $\frac{a\sigma^2}{b} = 0.00764116$ in (4) and solving (by using Mathematica 5.1), we get $e^{bt^*} = 1.2008$. For this particular $\frac{a\sigma^2}{b}$ and e^{bt^*}, we have $x = 0.00924616$, which is *greater than* 0.0077. Therefore we reduce the value of $\frac{a\sigma^2}{b}$, using (4), until x satisfies the inequality $x \leq 0.0077$. This results in

$$\frac{a\sigma^2}{b} \leq 0.0064. \tag{14}$$

Summing up, we have

Theorem: For a given a, b there exists $\sigma^2 > 0$ satisfying the estimation (14) provided $\frac{a\sigma^2}{b}$ is significantly equal to $ln(1 - \frac{a\sigma^2}{b})$.

Surprisingly, the numerical samples show that for a given $\frac{a\sigma^2}{b}$ the solution of equation (4),

$$e^{bt^*} \quad \text{remains in the neighbourhood of } 1.2.$$

Table 1 (reprinted from [3, 4, 8, 9, 11, 12])

Species	a	b	σ^2	$\frac{a\sigma^2}{b} < 0.0064$	e^{bt^*}	$\frac{a\sigma^2}{b} + ln(1 - \frac{a\sigma^2}{b}) = Error$
Mosquito	0.002	0.332	1.0	0.0060241	1.20059	-0.0000182181
Drosophila Melanogaster	0.0019	0.24	0.67	0.00530417	1.20033	-0.0000141171
Drosophila Melanogaster	0.00028	0.1608	0.60	0.00104478	1.19713	-0.000000546163
Callosobruchus Maculatus	0.002	0.261	0.518	0.00396935	1.20049	-0.00000789878
Human	0.0001	0.1	0.25	0.00025	1.20932	-0.0000000312552
Parage Ageria	0.0038	0.19	1.22	0.0244	1.20247	-0.000302613
Mosquito	0.003	0.2238	1.0	0.0134048	1.2011	-0.0000906544
Mosquito	0.00815	0.22	1.0	0.00370455	1.20367	-0.000703616
Mosquito	0.0221	0.14	1.0	0.157857	1.21677	-0.0139484
Mosquito	0.01168	0.22	1.5	0.0796364	1.20803	-0.00335007

A simple comparsion of $\frac{a\sigma^2}{b}$ given in (14) with the numerical data (Table 1) reveals that estimation (14) holds upto $\frac{a\sigma^2}{b}$ is significantly equal to $-ln(1-\frac{a\sigma^2}{b})$.

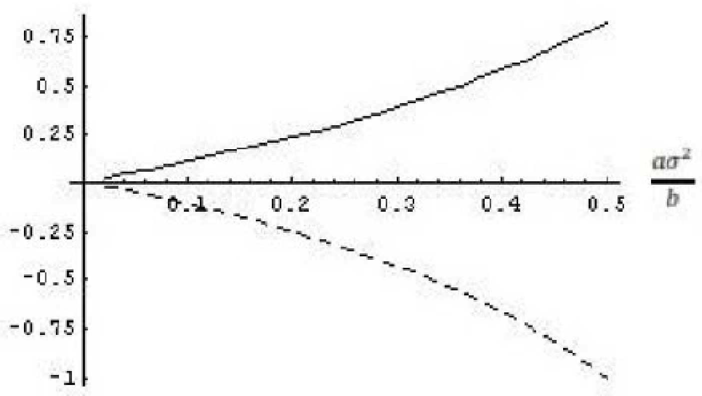

Figure 1:
Here the dotted curve represents $2\left[1-\frac{x}{(1+x)ln(1+x)}\right]$ and the another curve represents $Li_2(-x)$.

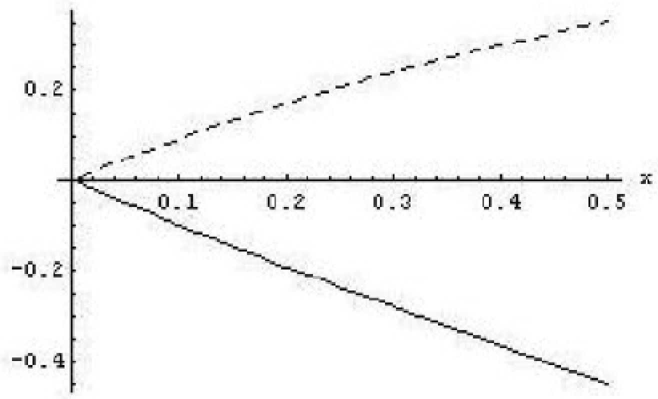

Figure 2:

Here the dotted curve represents $\frac{\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}}$ and the other curve represents $Li_2(-\frac{\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}})$.

3 CONCLUSION

Notice that considering the areawise approximation the estimation for $\frac{a\sigma^2}{b}$ significantly improved. It is worth mentioning that we are unable to get the estimation for e^{bt^*} from equation (4). Also we couldn't derive the equation for σ^2 alone.

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[16] $http : //www.intmath.com/Applications - integration/5_Centroid - area.php$

Appendix

4 Centroid of an area under the curve

By definition of centroid [16] we have

$$\overline{x} = \frac{total\ moments\ (x\ direction)}{total\ area} = \frac{\int\limits_a^b x(y_2 - y_1)dx}{\int\limits_a^b (y_2 - y_1)dx}$$

$$\overline{y} = \frac{total\ moments\ (y\ direction)}{total\ area} = \frac{\int\limits_c^d y(x_1 - x_2)dy}{\int\limits_c^d (x_1 - x_2)dy}.$$

For Gompetz Model

$$\overline{x} = \frac{\int\limits_a^b x(y_2 - y_1)dx}{\int\limits_a^b y_2 - y_1 dx}.$$

Here $a = 0$, $b = t^*$, $y_2 = ae^{bt}$, $y_1 = 0$.

$$\overline{x} = \frac{\int\limits_0^{t^*} ate^{bt}dt}{\int\limits_0^{t^*} ae^{bt}dt}$$

$$\overline{x} = \frac{e^{bt^*}(bt^* - 1) + 1}{b(e^{bt^*} - 1)}.$$

Similarly,

$$\overline{y} = \frac{\int\limits_c^d y(x_1 - x_2)dy}{\int\limits_c^d x_1 - x_2 dy}.$$

Here $c = 0$, $d = ae^{bt^*}$, $x_1 = t^*$, $x_2 = \frac{ln\frac{y}{a}}{b}$.

$$\overline{y} = \frac{\int\limits_0^{ce^{bt^*}} yt^*dy - \frac{\int\limits_0^{ae^{bt^*}} yln(\frac{y}{a})dy}{b}}{\int\limits_0^{ae^{bt^*}} t^*dy - \frac{\int\limits_0^{ae^{bt^*}} ln(\frac{y}{a})dy}{b}} = \frac{t^*\frac{(ae^{bt^*})^2}{2} - \frac{(ae^{bt^*})^2bt^*}{2b} - \frac{(ae^{bt^*})^2}{4b}}{t^*ae^{bt^*} - \frac{ae^{bt^*}.bt^*}{b} - \frac{ae^{bt^*}}{b}}.$$

After simplications we get

$$\overline{y} = \frac{ae^{bt^*}}{4}.$$

The centre of mass of A_2 is $(\frac{e^{bt^*}(bt^*-1)+1}{b(e^{bt^*}-1)}, \frac{ae^{bt^*}}{4})$

Logistic Frailty Model

$$\overline{x} = \frac{\int\limits_a^b x(y_2 - y_1)dx}{\int\limits_a^b y_2 - y_1 dx}.$$

Here $a = 0$, $b = t^*$, $y_2 = \frac{ae^{bt}}{1+\frac{a\sigma^2}{b}(e^{bt}-1)}$, $y_1 = 0$.

$$\overline{x} = \frac{\int\limits_0^{t^*} t\frac{ae^{bt}}{1+\frac{a\sigma^2}{b}(e^{bt}-1)}dt}{\int\limits_0^{t^*} \frac{ae^{bt}}{1+\frac{a\sigma^2}{b}(e^{bt}-1)}dt}.$$

Since $\int\limits_0^{t^*} t\frac{ae^{bt}}{1+\frac{a\sigma^2}{b}(e^{bt}-1)}dt = \frac{t^*ln(1+\frac{a\sigma^2}{b}(e^{bt^*}-1))}{\sigma^2} - \frac{\int\limits_0^{t^*} ln(1+\frac{a\sigma^2}{b}(e^{bt}-1))dt}{\sigma^2}$,
the above equation becomes

$$\overline{x} = \frac{\frac{t^*ln(1+\frac{a\sigma^2}{b}(e^{bt^*}-1))}{\sigma^2} - \frac{\int\limits_0^{t^*} ln(1+\frac{a\sigma^2}{b}(e^{bt}-1))dt}{\sigma^2}}{\frac{ln(1+\frac{a\sigma^2}{b}(e^{bt^*}-1))}{\sigma^2}},$$

which gives

$$\overline{x} = t^* - \frac{\int\limits_0^{t^*} ln(1 + \frac{a\sigma^2}{b}(e^{bt} - 1))dt}{ln(1 + \frac{a\sigma^2}{b}(e^{bt^*} - 1))}.$$

Similarly,

$$\overline{y} = \frac{\int\limits_c^d y(x_1 - x_2)dy}{\int\limits_c^d (x_1 - x_2)dy}.$$

Here $c = 0$, $d = \overline{\mu}(t^*) = \frac{ae^{bt^*}}{1+\frac{a\sigma^2}{b}(e^{bt^*}-1)}$, $x_1 = t^*$, $x_2 = \frac{ln(\frac{y}{a})+ln(\frac{1-\frac{a\sigma^2}{b}}{1-\frac{y\sigma^2}{b}})}{b}$.

$$\overline{y} = \frac{\int\limits_0^{\overline{\mu}(t^*)} y(t^* - \frac{ln(\frac{y}{a})-ln(\frac{1-\frac{a\sigma^2}{b}}{1-\frac{y\sigma^2}{b}})}{b})dy}{\int\limits_0^{\mu(t^*)} (t^* - \frac{ln(\frac{y}{a})+ln(\frac{1-\frac{a\sigma^2}{b}}{1-\frac{y\sigma^2}{b}})}{b})dy}. \quad (15)$$

Setting $\frac{y}{a} = z$, (15) gives

$$\overline{y} = \frac{\int\limits_0^{\frac{\overline{\mu}(t^*)}{a}} az(t^* - \frac{ln(z)+ln(\frac{1-z\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}})}{b})dz}{\int\limits_0^{\frac{\overline{\mu}(t^*)}{a}} (t^* - \frac{ln(z)+ln(\frac{1-z\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}})}{b})dz}.$$

Now consider the numerator $a \int\limits_0^{\frac{\overline{\mu}(t^*)}{a}} z(t^* - \frac{ln(z)}{b} + \frac{ln(\frac{1-z\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}})}{b})dz$.

Computing each integral seperately, we get

$\int\limits_0^{\frac{\overline{\mu}(t^*)}{a}} zt^*dz = \frac{\overline{\mu}^2(t^*)t^*}{2a^2}$,

$-\frac{\int\limits_0^{\frac{\overline{\mu}(t^*)}{a}} zln(z)dz}{b} = -\frac{-\overline{\mu}^2(t^*)}{2a^2b}\left[ln(\frac{\overline{\mu}(t^*)}{a}) - \frac{1}{2}\right]$, and $$\frac{\int_0^{\frac{\overline{\mu}(t^*)}{a}} ln(\frac{1-z\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}})dz}{b} = \frac{\frac{\overline{\mu}(t^*)}{2a^2} ln(\frac{1-\frac{\overline{\mu}(t^*)}{a}\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}}) + \frac{1}{2}(\frac{b}{a\sigma^2})^2(\frac{-\overline{\mu}^2(t^*)\sigma^4}{2b^2} - \frac{\overline{\mu}(t^*)\sigma^2}{b} - ln(1-\frac{\overline{\mu}(t^*)\sigma^2}{b}))}{b}.$$

Finally, we get

$$a\int_0^{\frac{\overline{\mu}(t^*)}{a}} z(t^* - \frac{ln(z)}{b} + \frac{ln(\frac{1-z\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}})}{b})dz - \frac{\overline{\mu}^2(t^*)t^*}{2a} - \frac{-\overline{\mu}^2(t^*)}{2ab}\left[ln(\frac{\overline{\mu}(t^*)}{a}) - \frac{1}{2}\right]$$
$$+ a\left(\frac{\frac{\mu(t^*)}{2a^2} ln(\frac{1-\frac{\overline{\mu}(t^*)}{a}\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}}) - \frac{1}{2}(\frac{b}{a\sigma^2})^2(\frac{-\mu^2(t^*)\sigma^4}{2b^2} - \frac{\mu(t^*)\sigma^2}{b} - ln(1 - \frac{\mu(t^*)\sigma^2}{b}))}{b}\right).$$

After some little algebra, we get

$$\text{Numerator} = \frac{1}{2}\frac{b}{a\sigma^4}\left[-\frac{\overline{\mu}(t^*)\sigma^2}{b} + bt^* - ln(1-\frac{a\sigma^2}{b}) + ln(e^{-bt^*} + \frac{a\sigma^2}{b}(1-e^{-bt^*}))\right].$$

Similarly consider the denominator $\int_0^{\frac{\overline{\mu}(t^*)}{a}} (t^* - \frac{ln(z)}{b} + \frac{ln(\frac{1-z\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}})}{b})dz$.

We compute each integral seperately.

$$\int_0^{\frac{\mu(t^*)}{a}} t^* dz = \frac{\overline{\mu}(t^*)t^*}{a},$$

$$-\frac{\int_0^{\frac{\overline{\mu}(t^*)}{a}} ln(z)dz}{b} = -\frac{-\overline{\mu}(t^*)}{ab}\left[ln(\frac{\overline{\mu}(t^*)}{a}) - 1\right],$$

and $$\frac{\int_0^{\frac{\overline{\mu}(t^*)}{a}} ln(\frac{1-z\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}})dz}{b} = \frac{\frac{\overline{\mu}(t^*)}{a} ln(\frac{1-\frac{\overline{\mu}(t^*)}{a}\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}}) + \frac{b}{a\sigma^2}(\frac{-\overline{\mu}(t^*)\sigma^2}{b} - ln(1 - \frac{\overline{\mu}(t^*)\sigma^2}{b}))}{b}.$$

$$\text{Denominator} = \frac{\overline{\mu}(t^*)t^*}{a} - \frac{-\overline{\mu}(t^*)}{ab}\left(ln(\frac{\overline{\mu}(t^*)}{a}) - 1\right) + \frac{\frac{\overline{\mu}(t^*)}{a} ln(\frac{1-\frac{\mu(t^*)}{a}\frac{a\sigma^2}{b}}{1-\frac{a\sigma^2}{b}}) + \frac{b}{a\sigma^2}(\frac{-\overline{\mu}(t^*)\sigma^2}{b} - ln(1-\frac{\mu(t^*)\sigma^2}{b}))}{b}.$$

After some little algebra, we get

$$\text{Denominator} = \frac{1}{a\sigma^2}\left[bt^* - ln(1 - \frac{a\sigma^2}{b}) + ln(e^{-bt^*} + \frac{a\sigma^2}{b}(1 - e^{-bt^*}))\right].$$

$$\text{Thus,} \quad \overline{y} = \frac{b}{2\sigma^2}\left[\frac{bt^* - ln(1 - \frac{a\sigma^2}{b}) + ln(e^{-bt^*} + \frac{a\sigma^2}{b}(1 - e^{-bt^*})) - \frac{\overline{\mu}(t^*)\sigma^2}{b}}{bt^* - ln(1 - \frac{a\sigma^2}{b}) + ln(e^{-bt^*} + \frac{a\sigma^2}{b}(1 - e^{-bt^*}))}\right]$$

or, equivalently

$$\overline{y} = \frac{b}{2\sigma^2}\left[1 - \frac{\frac{\overline{\mu}(t^*)\sigma^2}{b}}{bt^* - ln(1 - \frac{a\sigma^2}{b}) + ln(e^{-bt^*} + \frac{a\sigma^2}{b}(1 - e^{-bt^*}))}\right].$$

Since $-ln(1 - \frac{1-\overline{\mu}(t^*)\sigma^2}{b}) = bt^* - ln(1 - \frac{a\sigma^2}{b}) + ln(e^{-bt^*} + \frac{a\sigma^2}{b}(1 - e^{-bt^*}))$ the above equation reduces to

$$\overline{y} = \frac{b}{2\sigma^2}\left[1 - \frac{\frac{\overline{\mu}(t^*)\sigma^2}{b}}{-ln(1 - \frac{1-\overline{\mu}(t^*)\sigma^2}{b})}\right],$$

or, equivalently

$$\overline{y} = \frac{b}{2\sigma^2}\left[1 - \frac{x}{(1+x)ln(1+x)}\right], \text{where} \quad x = \frac{\frac{a\sigma^2}{b}e^{bt^*}}{1 - \frac{a\sigma^2}{b}}.$$

The centre of mass of A_1 is $\left(t^* - \frac{\int_0^{t^*} ln(1+\frac{a\sigma^2}{b}(e^{bt}-1))}{ln(1+\frac{a\sigma^2}{b}(e^{bt^*}-1))}, \frac{b}{2\sigma^2}\left[1 - \frac{x}{(1+x)ln(1+x)}\right]\right)$.