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Totally Sequential Cordial Graphs

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Abstract

Suppose G= (V, E) is a simple graph with vertex set V and edge set E. A vertex labeling f: V → {0, 1} induces an edge labeling f*: E →{0, 1} defined by f*(x y)= |f(x) - f(y)| . f is called a cordial labeling of G if the number of vertices labeled '0' and the number of vertices labeled '1' differ by atmost 1 and the number of edges labeled '0' and the number of edges labeled '1' differ by atmost 1. A graph with a cordial labeling is called a cordial graph. If the total number of vertices and edges labeled with '0' and the total number of vertices and edges labeled with '1' differ by atmost 1, it is called a Totally Sequential Cordial (TSC) labeling of G. A graph with a Totally Sequential Cordial labeling is called a Totally Sequential Cordial graph. In this paper we find the Totally Sequential Cordial labeling for certain graphs.

Keywords: Cordial labeling, simply sequential labeling, totally sequential cordial labeling.

2000 Mathematics Subject Classification: 05C78

Introduction

It is well known that graph theory has applications in many other fields. One area of considerable research potential on graph theory is that of graph labeling. An excellent reference on this is the survey by Gallian [6].

Two of the most important types of labelings are called graceful and harmonious labelings. Graceful labelings were introduced by Rosa [10] in 1996 and Golomb [7] in 1972. Harmonious labelings were first studied by Graham and Sloane [8] in 1980. A third type of labeling is cordial labeling and was introduced by Cahit [3] in 1990. Recently M. Sundaram, R. Ponraj, and S. Somasundaram [12] have introduced total product cordial graphs. P.Selvaraju and B. Nirmala Gnanam Pricilla [11] have proved that $P_m \times P_n$ for all m, n is a total product cordial graph. In 2002, Ibrahim Cahit [4] defined totally sequential cordial labeling, which is a weaker version of simply sequential labeling of graphs.

Suppose G= (V, E) is a simple graph. A vertex labeling f: V → {0, 1} induces an edge labeling f*: E → {0, 1} defined by f* (xy) = |f(x) – f(y)|. Let v_0 and v_1 be the number of vertices labeled with '0'and '1' respectively. Let e_0 and e_1 be the number of edges labeled with '0', and '1' respectively. Such a labeling is cordial if both $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$. A graph is called a cordial graph if it has a cordial labeling. Let $t_0= v_0 + e_0$ and $t_1 = v_1 + e_1$. If $|t_0 - t_1| \leq 1$ then the labeling is called a TSC labeling. A graph with a TSC labeling is called a TSC graph.

A.T. Diab [9] has proved that the join of the path P_n and the star $K_{1, m}$ is cordial for all n and all m if and only if (n, m) ≠ (2, 1). Also he proved that the union of path P_n and the star $K_{1, m}$ is cordial for all n and all m if and only if (n, m) ≠ (2, 1). But, this condition fails when n=2 and m>1 is odd. That is $P_2+K_{1, m}$ and $P\cup K_{1, m}$ where m is odd are not cordial.

Cahit [8] proved that every cordial graph is TSC, C_n is TSC for all n > 2, trees are TSC, the wheel W_n is TSC for all n > 3. He gave some conditions for a complete graph K_n to be TSC. But these conditions are contradicted by some complete graphs K_6, K_{13}, K_{22}, K_{33}, K_{46} and so on.

In this paper, we prove that the graphs C_n, and P_n are TSC graphs. Moreover, we show that the join of the path P_n and the star $K_{1, m}$ is TSC if and only if n ≠ 2 and m is even; the union of the path P_n and the star $K_{1, m}$ is total cordial if and only if n ≠ 2 and m is even. Also we modify the conditions for K_n to be TSC, which Cahit has already established.

Terminologies and notations

We introduce some terminologies and notations for a graph with 4r vertices. Let L_{4r} denote the labeling 00110011… 0011, S_{4r} denote the labeling 11001100…1100 and M_{2r} denote the labeling 0101… 0101. Also these are modified by adding symbols at one end or other (or both). Thus $01L_{4r}$ denotes 0100110011…0011.

For a given labeling of the graphs G and H, let x_i, a_i (for i= 0, 1) denote the number of vertices and edges of G labeled with 'i' respectively. Let y_i, b_i be the number of vertices and edges of H labeled with 'i' respectively.

It follows that $v_0 = x_0 + y_0;$ $v_1 = x_1 + y_1;$ $e_0 = a_0 + b_0 + x_0y_0 + x_1 y_1;$ $e_1 = a_1 + b_1 + x_0y_1 + x_1y_0$. Thus $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1) (y_0 - y_1)$; and $t_0 - t_1 = (v_0 - v_1) + (e_0 - e_1)$.

Totally sequential cordial labeling for cycles and paths

Theorem 3.1. The cycle graph C_n is a totally sequential cordial graph for all n>2.

Proof. We denote by C_n the graph consisting of a cycle with n points. Cahit showed that a unicyclic graph is cordial unless it is C_{4k+2}.

Let n=4r+i, where i=0, 1, 2, 3 and for some r ∈ N. The TSC labeling we use for C_n is given in the following table.

n=4r+i i=0, 1, 2, 3	Labeling of C_n	v_0	v_1	e_0	e_1	v_0 - v_1	$e_0 – e_1$	$t_0 – t_1$
i = 0	L_{4r}	2r	2r	2r	2r	0	0	0
i = 1	$1L_{4r}$	2r	2r+1	2r+1	2r	-1	1	0
i = 2	$11L_{4r}$	2r	2r+2	2r+2	2r	-2	2	0
i = 3	$L_{4r}001$	2r+2	2r+1	2r+1	2r+2	1	-1	0

The last column of this table shows that $|t_0 – t_1| = 0 \leq 1$. Hence C_n is TSC for all n>2.

Theorem3.2. The path graph P_n is a totally sequential cordial graph for all n.

Proof. We denote by P_n a path with n points. The total sequential cordial labeling that we use are given as follows. Let n=4r+i, where i= 0, 1, 2, 3 and for some natural number 'r'.

n=4r+i, i=0, 1, 2, 3	Labeling of P_n	v_0	v_1	e_0	e_1	v_0 - v_1	$e_0 – e_1$	$t_0 – t_1$
i= 0	L_{4r}	2r	2r	2r	2r - 1	0	1	1
i=1	$1L_{4r}$	2r	2r+1	2r	2r	-1	0	-1
i= 2	$01L_{4r}$	2r+1	2r+1	2r	2r+1	0	-1	-1
i=3	$001L_{4r}$	2r+2	2r+1	2r+1	2r+1	1	0	1

From the last column we observe that, $|t_0 – t_1| \leq 1$. Hence P_n is TSC for all n.

Theorem 4. The complete graph K_4 is not totally sequential cordial.

Proof. All possible vertex labelings satisfying the condition that $v_0 = 1$ and $v_1 = 3$ are [0111], [1101], [1110].

All possible vertex labelings with $v_0 = 3$ and $v_1 = 1$ are [0001], [0010], [0100], [1000].

For all the above labelings, $e_0 =3$ and $e_1 = 3$.

Therefore $|t_0 – t_1| =2$ not less than or equal to 1.

All possible vertex labelings with $v_0 = v_1 = 2$ are [0011], [1001], [1100], [0110], [0101], [1010].

For these labelings $e_0 = 2$ and $e_1 = 4$.

Therefore $t_0 - t_1 = 0 - 2 = -2$

Hence K_4 is not totally sequential cordial.

Join and union of Paths and Stars

Definition. Let G_1 and G_2 be the graphs having disjoint point sets V_1 and V_2 and line sets X_1 and X_2 respectively. Their join denoted by G_1+G_2 is a graph with point set $V=V_1 \cup V_2$ and line set $X=X_1 \cup X_2$ together with all lines joining V_1 and V_2.

The union of G_1 and G_2 denoted by $G_1 \cup G_2$ is a graph with point set $V=V_1 \cup V_2$ and line set $X=X_1 \cup X_2$.

Theorem 5.1. The join of the path P_n and the star $K_{1,\,m}$ is TSC for all $n \neq 2$ and all even m

Proof. Let $A_0=L_{4r}$, $A_1=L_{4r}0$ =00110011…00110, $A_2=L_{4r}01$, $A_2'=L_{4r}10$, $A_3=L_{4r}001$, $B_0=1M_{2s}$=10101…01, $B_0'=0M_{2s}$, $B_1=01M_{2s}$.

Let n=4r+i, where i=0, 1, 2, 3and m=2s+j, where j=0, 1 and r, s ∈ N.

Table: 5.1.1

n=4r+i, i=0, 1, 2, 3	Labeling of P_n	x_0	x_1	a_0	a_1
i= 0	A_0	2r	2r	2r	2r – 1
i= 1	A_1	2r +1	2r	2r	2r
i=2	A_2	2r+1	2r+1	2r	2r+1
i=2, r≠0	A_2'	2r+1	2r+1	2r+1	2r
i=2, r=0	A_2'	2r+1	2r+1	2r	2r+1
i=3	A_3	2r+2	2r+1	2r+1	2r+1

Table: 5.1.2

m=2s+j, j=0, 1	Labeling of $K_{1,\,m}$	y_0	y_1	b_0	b_1
j=0	B_0	s	s+1	s	s
	B_0'	s+1	s	s	s
j=1	B_1	s+1	s+1	s	s+1

By using the labelings of P_n and $K_{1,\,m}$ from tables 5.1.1 & 5.1.2 and by using the formula $v_0\text{-}v_1 = (x_0\text{-}x_1) + (y_0\text{-}y_1)$; $e_0\text{-}e_1 = (a_0\text{-}a_1) + (b_0\text{-}b_1) + (x_0\text{-}x_1)(y_0\text{-}y_1)$.

Then the values of $t_0\text{-}t_1$ are calculated.

Table: 5.1.3 (Labelings of P_n+$K_{1, m}$)

n=4r+i, i=0, 1, 2, 3	m=2s+j, j=0, 1	Labeling of P_n	Labeling of $K_{1, m}$	$v_0 - v_1$	$e_0 - e_1$	$t_0 - t_1$
i=0	0	A_0	B_0	-1	1	0
i=1	0	A_1	B_0	0	-1	-1
i=2	0	A_2	B_0'	1	-1	0
i=3	0	A_3	B_0	0	-1	-1
i=0	1	A_0	B_1	0	0	0
i=1	1	A_1	B_1	1	-1	0
i=2, r=0	1	A_2 or A_2'	B_1	0	-2	-2
i=2, r≠0	1	A_2'	B_1	0	0	0
i=3	1	A_3	B_1	1	-1	0

From the last column of table 5.1.3, we can easily observe that $|t_0 - t_1| \leq 1$, when n≠2 and m is even.

Also when n=2, all possible labelings of P_n are [0 0], [0 1], [1 1]

Labeling of $K_{1, m}$ where m is odd is B_1 . These labelings give us $v_0 - v_1 = 0$, $e_0 - e_1 = \pm 2$ or $v_0 - v_1 = \pm 2$, $e_0 - e_1 = 0$.

In either of these cases, $|t_0 - t_1| = 2$.

Hence, $P_n + K_{1, m}$ is TSC iff $n \neq 2$ and m is even.

Theorem 5.2. The union of the path P_n and the star $K_{1, m}$ is totally sequential cordial for all n except 2 and all even m.

Proof. By using the labelings of the path P_n in table 5.1.1 and the star $K_{1, m}$ in the table 5.1.2 and using the formula $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, we can compute the values.

Table 5.2.1 (Labeling of $P_n \cup K_{1, m}$)

n=4r+i, i=0, 1, 2, 3	m=2s+j, j=0, 1	Labeling of P_n	Labeling of $K_{1, m}$	$v_0 - v_1$	$e_0 - e_1$	$t_0 - t_1$
i=0	0	A_0	B_0	-1	1	0
i=1	0	A_1	B_0	0	0	0
i=2	0	A_2	B_0'	1	-1	0
i=3	0	A_3	B_0	0	0	0
i=0	1	A_0	B_1	0	0	0
i=1	1	A_1	B_1	1	-1	0
i=2, r≠0	1	A_2'	B_1	0	0	0
i=2, r=0	1	A_2 or A_2'	B_1	0	-2	-2
i=3	1	A_3	B_1	1	-1	0

From the last column of table 5.2.1, we can easily observe that $|t_0 - t_1| \leq 1$, when $n \neq 2$ and m is even.

Also when n = 2, all possible labelings of P_n are [0 0], [0 1], [1 1].

When m is odd, labelings of $K_{1,m}$ are B_1 .

By using the labelings we compute $v_0 - v_1$, $e_0 - e_1$ and $t_0 - t_1 = 2$ or -2.

Hence $P_n \cup K_{1,m}$ is TSC iff $n \neq 2$ and m is even.

Theorem 6: The complete graph K_n is totally sequential cordial if and only if

1. $\sqrt{n+1}$ is an integer, when $n \equiv 0 (\text{mod } 4)$
2. $\sqrt{\frac{n-1}{4}}$ or $\sqrt{\frac{n+3}{4}}$ is an integer, when $n \equiv 1 (\text{mod } 4)$
3. $\sqrt{n-1}$ or $\sqrt{n+3}$ is an integer, when $n \equiv 2 (\text{mod } 4)$
4. $\sqrt{\frac{n+1}{4}}$ is an integer, when $n \equiv 3 (\text{mod } 4)$

Proof. The complete graph K_n has n vertices and nC_2 edges. Let f be a total sequential cordial labeling of K_n. Then $| t_0 - t_1 | \leq 1$.

Also $t_0 + t_1 = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2} = \begin{Bmatrix} even\ if\ n \equiv 0, 3(mod 4) \\ odd\ if\ n \equiv 1, 2(mod\ 4) \end{Bmatrix}$. We consider two cases.

Case 1: Let $t_0 + t_1$ be even.
Then $n \equiv 0$, 3(mod 4).Since f is a TSC labeling of K_n, $t_0 = t_1$.

Under this labeling f, the complete graph K_n can be decomposed as: $K_n = K_p \cup K_r \cup K_{p,r}$, where K_p is the sub-complete graph of K_n whose vertices are labeled with 1's; K_r is the sub-complete graph of K_n whose vertices are labeled with 0's and $K_{p,r}$ is the complete bipartite sub-graph of K_n with the bipartition $V(K_p) \cup V(K_r)$ which its edges labeled with all 1's.

Then n= p + r.

Clearly, for the labeling f, we write

$t_1 = p + rp$ and $t_0 = pC_2 + r + rC_2$.

Using $t_0 = t_1$, we get, $(r-p)^2 - 3p + r = 0$.

Put, p = n – r in the above equation, we get

$4r^2 - 4(n-1)r + n^2 - 3n = 0$

Solving this for r, we obtain $r_{1,2} = \frac{(n-1) \pm \sqrt{n+1}}{2}$

This $r_{1,2}$ will represent the order of the sub-complete graph K_r . We can easily see that, for $n \equiv 0$ (mod 4), K_n is TSC iff $\sqrt{n+1}$ is an integer. Also, for $n \equiv 3$ (mod 4), K_n is TSC iff $\sqrt{\frac{n+1}{4}}$ is an integer.

Case 2*:* Let $t_0 + t_1$ be odd.
Clearly $n \equiv 1$, 2(mod 4)

Since f is a TSC labeling of K_n, there arise two cases $t_1 > t_0$ or $t_0 > t_1$.

Sub case 2. (i): Assume that $t_1 > t_0$.
Then $t_1 = t_0 + 1$.

By the same decomposition of case 1, we write,
$t_1 = p + rp$ and $t_0 = pC_2 + r + rC_2$ ___________ (1)
Also, $p = n - r$ ___________ (2)

Using all these equations, we get the quadratic equation,
$4r^2 - 4(n-1)r + n^2 - 3n + 2 = 0$

Solving this for r we get,
$r_{1,2} = \frac{(n-1)\pm\sqrt{(n-1)}}{2}$

This gives the order of the sub-complete graph k_r. In order to have integer values for $r_{1,\,2}$, for n≡1(mod 4) and n≡2(mod 4) respectively $\sqrt{\frac{n-1}{4}}$ and $\sqrt{n-1}$ must be an integer.

Sub case 2. (ii): Let $t_0 > t_1$.
We can take $t_0 = t_1 + 1$ ___________ (3)
By using equations (1) and (2), equation (3) becomes
$4r^2 - 4(n - 1)r + n^2 - 3n - 2 = 0$

Solving this for r we get,
$r_{1,2} = \frac{(n-1)\pm\sqrt{(n+3)}}{2}$

This gives the order of the sub-complete graph k_r. It can easily be seen that,
when n≡1(mod 4), K_n is TSC iff $\sqrt{\frac{n+3}{4}}$ is an integer and
when n ≡2(mod 4), K_n is TSC iff $\sqrt{n+3}$ is an integer.

Thus from case (2) we observe that
K_n is TSC iff $\sqrt{\frac{n-1}{4}}$ or $\sqrt{\frac{n+3}{4}}$ is an integer, when n≡1(mod 4)
And for n ≡2(mod 4), K_n is TSC iff $\sqrt{n-1}$ or $\sqrt{n+3}$ is an integer.
Hence the theorem.

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