Strong Convergence of Iterative Algorithms for Nonexpansive Mappings

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Abstract

In this paper, strong convergence of implicit iterates generated by a nonexpansive mapping in a real uniformly smooth and uniformly convex Banach spaces are proved with mild assumptions on control conditions. The results presented in this paper are extensions and improvements of the corresponding results in the literature.

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Introduction

Let $K$ be a nonempty closed convex subset of a real Banach space $X$. Let $T : K \rightarrow K$ be a mapping. We use $Fix(T)$ to denote the set of fixed points of $T$; that is, $Fix(T) = \{x \in K : T(x) = x\}$.

Recall that $T$ is a contraction mapping if there exists a constant $\lambda \in [0, 1)$ such that $\|Tx - Ty\| \leq \lambda \|x - y\|$ for each $x, y \in K$. By Banach contraction mapping principle if $T$ is a contraction mapping then $T$ has a unique fixed point in $K$ and for any $x_0 \in K$ the Picard iteration, $x_{n+1} = Tx_n = T^{n+1}x_0, n \geq 0$ converges strongly to the fixed point of $T$.

The mapping $T$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in K$. For nonexpansive mappings existence of fixed points is not guaranteed. Even if it exists it is not unique and we cannot approximate it by Picard iteration. Thus it is necessary to search other iterative methods to approximate fixed points of
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nonexpansive mappings in Banach spaces. Nowadays iterative methods of approximating fixed points of nonexpansive mappings and its generalizations have received much interest of researchers. To list some of them; see [1]-[8], [10], [12] - [41], and the references there in.

In 1953, Mann[17] introduced an iterative algorithm which is now referred to as the Mann iterative algorithm which is defined explicitly as

\[ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T x_n, \quad n \geq 0, \]  

where \( x_0 \in K \) is chosen arbitrarily, and \( \{\alpha_n\} \) is a sequence in \([0, 1]\) which satisfies certain control conditions. In the last half century, the convergence of Mann iteration was investigated by different authors in Banach spaces with wide range of geometric properties; see [3].

Recently so many authors modified Mann iteration to obtain strong convergence to a fixed point of \( T \). We see some of them in the next paragraphs which are related to our main result.

In 2001, Xu and Ori [34] introduced Mann-type implicit iteration defined as

\[ x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T x_n, \quad n \geq 1, \]  

where \( x_0 \in K \) is chosen arbitrarily and \( \{\alpha_n\} \) is a sequence in \([0, 1]\) that satisfies certain control conditions. They proved weak convergence of the iteration in Hilbert space setting.

In 2004, Osilike[21] proved weak convergence of the iteration in (1.2) for the class of strictly pseudocontractive self-mapping \( T \) of \( K \) in Hilbert space.

In 2005, Chidume and Shahzad [6] proved strong convergence of the iteration in (1.2) in more general Banach space by imposing compactness like condition on nonexpansive mapping.

Other group of authors focused on the Halpern [10] type approach to get strong convergence of iterations. These researchers were concentrated on weakening the conditions on parameters imposed by Halpern. In 2007, Yao et al [38] proved strong convergence of modified Mann iteration defined implicitly as

\[ x_n = \alpha_n u + \beta_n x_{n-1} + \gamma_n T x_n, \quad n \geq 1, \]  

where \( x_0 \in K \) arbitrary, \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) are sequences in \([0, 1]\) which satisfy certain control conditions, and \( T \) is pseudocontractive mapping in some family of Banach spaces. In this line, Yao et al [39] investigated explicit Mann-type iteration defined as

\[ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n, \quad n \geq 0. \]  

They proved that the explicit iteration defined in (1.4) converges strongly to a fixed point of a nonexpansive mapping \( T \) with some control conditions.

In 2008, Zhao et al [40] introduced more general implicit iteration defined as

\[ x_n = \alpha_n x_{n-1} + \beta_n T x_{n-1} + \gamma_n T x_n, \quad n \geq 1, \]  

where \( \alpha_n, \beta_n, \gamma_n \) are sequences in \([0, 1]\) which satisfy certain control conditions. They proved that the implicit iteration defined in (1.5) converges strongly to a fixed point of a nonexpansive mapping \( T \) with some control conditions.
where $x_0 \in K$ arbitrary, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0, 1]$. They proved:

**Theorem 1.1.** ([40], Theorem 2.1) Let $X$ be a real uniformly convex Banach space which satisfies Opial’s condition, let $K$ be a nonempty closed convex subset of $X$, and suppose $T : K \to K$ is a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ and $0 < a \leq \gamma_n \leq b < 1$, for each $n \geq 1$, where $a$, $b$ are some constants. Then implicit iteration $\{x_n\}_{n=1}^{\infty}$ defined by (1.5) converges weakly to a fixed point of $T$.

**Theorem 1.2.** ([40], Theorem 2.2) Let $K$ be a nonempty closed convex subset of a real uniformly convex Banach space $X$ and suppose $T : K \to K$ is a nonexpansive mapping with a nonempty fixed point set $\text{Fix}(T)$. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0, 1]$ that satisfy those conditions of Theorem 1.1 for some constants $a$, $b$. Let $x_0 \in K$ be fixed. If $T$ is semi-compact then implicit iteration $\{x_n\}_{n=1}^{\infty}$ defined in (1.5) converges strongly to a fixed point of $T$.

The second author already investigated that the iteration defined in (1.5) converges weakly to a fixed point of $T$ in uniformly convex Banach space. So that in Theorem 1.1 Opial’s condition overflows. Strong convergence was also proved by imposing compactness assumption on the mapping.

In this paper our main target is to obtain strong convergence by modifying the iteration defined in (1.5) without assuming compactness on the mapping.

**Preliminaries**

A real Banach space $X$ is said to be *uniformly convex* [9] if for each $\epsilon > 0$ there is $\delta > 0$ such that for each $x, y \in X$

$$
\|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \Rightarrow \|x + y\| \leq 2(1 - \delta).
$$

The function

$$
\delta_X(\epsilon) = \inf \{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\}, \epsilon \in [0, 2]
$$

is said to be *modulus of convexity of $X$*. Thus $X$ is uniformly convex if $\delta_X(\epsilon) > 0, \forall \epsilon > 0$.

Let $X$ be a real Banach space. Let $S_X = \{x \in X : \|x\| = 1\}$ denote the unit sphere of $X$. The norm on $X$ is said to be *Gâteaux differentiable* if the limit

$$
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
$$

(2.1)
exists for each $x, y \in S_X$ and in this case $X$ is said to be smooth. If for each $y \in S_X$ the limit (2.1) is attained uniformly for $x \in S_X$, $X$ is said to have a uniformly Gâteaux differentiable norm. If the limit (2.1) exists uniformly for $(x, y) \in S_X \times S_X$ then $X$ is is said to be uniformly smooth.

Let $X$ be a real Banach space with dual $X^*$. The normalized duality mapping $J : X \to 2^{X^*}$ is defined

$$J(x) = \{ x^* \in X^* : (x, x^*) = \|x\|^2, \|x^*\| = \|x\| \}, x \in X,$$

where $(..)$ denotes the pairing between $X$ and $X^*$, i.e. $(x, x^*) = x^*(x)$. It is known that if $X$ is uniformly smooth Banach space then the duality mapping $J$ is single valued and norm to weak star uniformly continuous on bounded sets of $X$. For more on normalized duality mapping, we refer [11].

The following lemmas are key tools in the proof of our main result of the paper.

**Lemma 2.1.** ([23]) Let $K$ be a nonempty closed convex subset of a real uniformly smooth Banach space $X$ and suppose $T : K \to K$ is a nonexpansive mapping such that $Fix(T) \neq \emptyset$. Let $u \in K$ be fixed. For every $t \in (0, 1)$, let $y_t \in K$ denote the unique fixed point of the contraction mapping $T_u$ given by $T_u(x) = tu + (1 - t)T(x), x \in K$. Then $\{y_t\}$ converges strongly as $t \to 0$ to $p \in Fix(T)$ which is nearest to $u$.

**Lemma 2.2.** ([31]) Let $X$ be a real Banach space with the normalized duality mapping $J$. Then, for each pair $x, y \in X$ and for each $j(x + y) \in J(x + y)$, the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, j(x + y) \rangle.$$

**Lemma 2.3.** ([8], Lemma 7) Let $p > 1, r > 0$ be two fixed numbers. Then a Banach space $X$ is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$, such that

$$\|ax + by + c\|^p \leq a \|x\|^p + b \|y\|^p + c \|z\|^p - \max\{w_p(a, b), w_p(a, c), w_p(b, c)\} g(\|x - y\|),$$

for all $x, y, z \in B(O, r) = \{x \in X : \|x\| \leq r\}$ and $a, b, c \in [0, 1]$ with $a + b + c = 1$. 


Lemma 2.4. ([32]) Let \( \{\lambda_n\} \subseteq [0, \infty), \{\alpha_n\} \subseteq (0, 1) \) and \( \{\beta_n\} \subseteq (-\infty, \infty) \) be real sequences satisfying

1. \( \lambda_{n+1} \leq (1 - \alpha_n)\lambda_n + \alpha_n\beta_n, \)
2. \( \sum_{n=1}^{\infty} \alpha_n = +\infty, \) and
3. \( \limsup_{n \to \infty} \beta_n \leq 0 \) or \( \sum_{n=1}^{\infty} \alpha_n\beta_n < +\infty. \)

Then \( \lim_{n \to \infty} \lambda_n = 0. \)

Main Results

Let \( K \) be a nonempty closed convex subset of a real Banach space \( X. \) Let \( T : K \to K \) be a nonexpansive mapping such that \( \text{Fix}(T) \neq \emptyset. \)

Let \( \alpha, \beta, \gamma, \) and \( \delta \) be real numbers in \([0, 1]\) such that \( \alpha + \beta + \gamma + \delta = 1. \)

Then for any two fixed \( u, z \in K, \) the mapping \( T_z : K \to K \) given by

\[
T_zx = \alpha u + \beta z + \gamma Tz + \delta Tx, \quad x \in K,
\]

satisfies

\[
\|T_zx - T_zy\| \leq \delta \|x - y\|,
\]

for all \( x, y \in K. \) Hence \( T_z \) is a contraction mapping if \( \delta < 1. \) By the Banach Contraction Mapping Principle, \( T_z \) has a unique fixed point, say \( x_z; \) that is,

\[
x_z = \alpha u + \beta z + \gamma Tz + \delta Tx_z.
\]

Motivated by the above fact, we introduce implicit iteration which extends and improves the iterations introduced by Zhao et al [40], Yao et al [38] and Yao et al [39] as follows. Let \( \{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\gamma_n\}_{n=1}^{\infty} \) and \( \{\delta_n\}_{n=1}^{\infty} \) be sequences in \([0, 1]\) such that \( \alpha_n + \beta_n + \gamma_n + \delta_n = 1, \) for each \( n \geq 1. \) Let \( u \in K \) be fixed. For given \( x_0 \in K \) arbitrarily, there is a unique sequence \( \{x_n\}_{n=1}^{\infty} \) defined implicitly as

\[
x_n = \alpha_n u + \beta_n x_{n-1} + \gamma_n T_n x_{n-1} + \delta_n T x_n, \quad n \geq 1,
\]

Remark 3.1. The implicit iteration defined in (3.1) is more general than iterations defined in (1.1), (1.2), (1.3), (1.4) and (1.5).
Consider that for each \( n \geq 1 \) and \( t \in (0, 1) \), the mapping \( T_{t,n} : K \rightarrow K \) given by
\[
T_{t,n}x = \frac{(1-\alpha_n)t}{\delta_n + t(\beta_n + \gamma_n)} u + \frac{(1-t)\delta_n}{\delta_n + t(\beta_n + \gamma_n)} Ty_{t,n}
\]
is a contraction. Let \( y_{t,n} \) denote the unique fixed point of \( T_{t,n} \); i.e.,
\[
y_{t,n} = \frac{(1-\alpha_n)t}{\delta_n + t(\beta_n + \gamma_n)} u + \frac{(1-t)\delta_n}{\delta_n + t(\beta_n + \gamma_n)} Ty_{t,n}.
\]
(3.2)

The equation in (3.2) is rewritten as
\[
y_{t,n} = tu + (1-t)[\frac{\beta_n + \gamma_n}{1-\alpha_n} y_{t,n} + \frac{\delta_n}{1-\alpha_n} Ty_{t,n}].
\]
(3.3)

If \( X \) is uniformly smooth, it follows from Lemma 2.1 that for each \( n \geq 1 \), we have
\[
\lim_{t \to 0} y_{t,n} = p_n \in \text{Fix}(T),
\]
which is nearest to \( u \). Therefore, for some \( p \in \text{Fix}(T) \), \( p_n = p_m = p \), for all \( n, m \geq 1 \).

The following main result of the paper shows that the implicit iteration \( \{x_n\} \) defined in (3.1) converges strongly to \( p \).

**Theorem 3.2.** Let \( K \) be a nonempty closed convex subset of a real uniformly smooth and uniformly convex Banach space \( X \) and suppose \( T : K \rightarrow K \) is a nonexpansive mapping with a nonempty fixed point set \( \text{Fix}(T) \). Let \( \{\alpha_n\}_{n=1}^{\infty} \), \( \{\beta_n\}_{n=1}^{\infty} \), \( \{\gamma_n\}_{n=1}^{\infty} \) and \( \{\delta_n\}_{n=1}^{\infty} \) be sequences in \((0, 1)\) such that
\begin{align*}
(1) \quad \alpha_n + \beta_n + \gamma_n + \delta_n &= 1, \quad \text{for each } n \geq 1; \\
(2) \quad \lim_{n \to \infty} \alpha_n &= 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = +\infty; \\
(3) \quad \lim_{n \to \infty} \delta_n &= 0.
\end{align*}
For fixed \( u \in K \) and for any initial point \( x_0 \in K \), the implicit iteration \( \{x_n\}_{n=1}^{\infty} \) defined in (3.1) converges strongly to a unique fixed point of \( T \) which is nearest to \( u \).
**Proof.** Let \( z \in Fix(T) \). Then for each \( n \geq 1 \) we get
\[
\|x_n - z\| \leq \alpha_n \|u - z\| + \beta_n \|x_{n-1} - z\| + \gamma_n \|Tx_{n-1} - z\| + \delta_n \|Tx_n - z\|
\]
so that
\[
\|x_n - z\| \leq \frac{\alpha_n}{1 - \delta_n} \|u - z\| + \frac{\beta_n + \gamma_n}{1 - \delta_n} \|x_{n-1} - z\|
\leq \max \{ \|u - z\|, \|x_{n-1} - z\| \}
\]
By induction we can obtain that
\[
\|x_n - z\| \leq \max \{ \|u - z\|, \|x_0 - z\| \}.
\]
Therefore \( \{x_n\} \) is a bounded sequence and thus \( \{Tx_n\} \) is also bounded. Now for each \( n \geq 1 \) it follows from (3.3) that
\[
y_{t,n} - x_n = t(u - x_n) + (1 - t)\left[\frac{\beta_n + \gamma_n}{1 - \alpha_n} (y_{t,n} - x_n) + \frac{\delta_n}{1 - \alpha_n} (Ty_{t,n} - x_n)\right].
\]
\[
\|y_{t,n} - x_n\|^2 \leq (1 - t)^2 \left[\frac{\beta_n + \gamma_n}{1 - \alpha_n} \|y_{t,n} - x_n\| + \frac{\delta_n}{1 - \alpha_n} \|Ty_{t,n} - x_n\|\right]^2 + 2t(u - x_n, J(y_{t,n} - x_n))
\leq (1 - t)^2 \left[\frac{\beta_n + \gamma_n}{1 - \alpha_n} \|y_{t,n} - x_n\| + \frac{\delta_n}{1 - \alpha_n} \|Ty_{t,n} - x_n\|\right]^2 + 2t(u - x_n, J(y_{t,n} - x_n))
\leq (1 - t)^2 \left[\|y_{t,n} - x_n\| + \frac{\delta_n}{1 - \alpha_n} \|Ty_{t,n} - x_n\|^2\right] + 2t(u - x_n, J(y_{t,n} - x_n))
\leq (1 - t)^2 \left[\|y_{t,n} - x_n\|^2 + \frac{\delta_n}{1 - \alpha_n} \|Ty_{t,n} - x_n\|^2\right] + 2t(u - x_n, J(y_{t,n} - x_n))
\leq (1 + t^2) \|y_{t,n} - x_n\|^2 + (1 - t)^2 \left[\frac{\delta_n}{1 - \alpha_n} \|Ty_{t,n} - x_n\|^2\right]
\leq (1 - t)^2 \left[\delta_n \|y_{t,n} - x_n\| + \frac{\delta_n}{1 - \alpha_n} \|Ty_{t,n} - x_n\|\right] + 2t(u - x_n, J(y_{t,n} - x_n)),
\]
which implies that
\[
\langle u - y_{t,n}, J(x_n - y_{t,n}) \rangle \leq \frac{t}{2} \|y_{t,n} - x_n\|^2 + \frac{(1 - t)^2 \delta_n}{2(1 - \delta_n)} \|y_{t,n} - x_n\| \|Ty_{t,n} - x_n\| + \|Tx_n - x_n\|^2.
\] (3.4)
Since \( \{x_n\}_{n=1}^{\infty}, \{y_{t,n} : t \in (0, 1), n = 1, 2, \ldots\} \) and \( \{Tx_n\}_{n=1}^{\infty} \) are bounded there exists a nonnegative real number \( r \) such that
\[
(x_n) : n = 0, 1, 2, \ldots \}
\]
where \( B(O, r) = \{x \in X : \|x\| \leq r\} \). Now it follows from (3.4) and (3.5) that
\[
\langle u - y_{t,n}, J(x_n - y_{t,n}) \rangle \leq 2t + \frac{(1 - t)^2 \delta_n}{t(1 - \delta_n)} r^2.
\] (3.6)
Letting \( n \to \infty \) in (3.6) we obtain
\[
\lim_{n \to \infty} \sup_{t \to 0} \langle u - y_{t,n}, J(x_n - y_{t,n}) \rangle \leq 4t r^2. \tag{3.7}
\]

Letting \( n \to \infty \) in (3.7) we get
\[
\lim_{t \to 0} \lim_{n \to \infty} \sup_{t \to 0} \langle u - y_{t,n}, J(x_n - y_{t,n}) \rangle \leq 0. \tag{3.8}
\]

Since the order of \( \limsup_{t \to 0} \) and \( \limsup_{n \to 1} \) is exchangeable in (3.8), we have
\[
\lim_{n \to \infty} \sup_{t \to 0} \langle u - p, J(x_n - p) \rangle \leq 0. \tag{3.9}
\]
where \( p = \lim_{t \to 0} y_{t,n} \) is a fixed point of \( T \) nearest to \( u \).

We see that for each \( n \geq 1 \)
\[
x_n - p = \alpha_n(u - p) + \beta_n(x_{n-1} - p) + \gamma_n(T(x_{n-1}) - p) + \delta_n(T(x_n - p)). \tag{3.10}
\]

Applying Lemma 2.2 and Lemma 2.3 to (3.10), we get
\[
\|x_n - p\|^2 \leq \|\beta_n(x_{n-1} - p) + \gamma_n(T(x_{n-1}) - p) + \delta_n(T(x_n - p))\|^2 + 2\alpha_n \langle u - p, J(x_n - p) \rangle \\
\leq (\beta_n + \gamma_n) \|x_{n-1} - p\|^2 + \delta_n \|x_n - p\|^2 + 2\alpha_n \langle u - p, J(x_n - p) \rangle,
\]
from which
\[
\|x_n - p\|^2 \leq \frac{\beta_n + \gamma_n}{1 - \delta_n} \|x_{n-1} - p\|^2 + 2\frac{\alpha_n}{1 - \delta_n} \langle u - p, J(x_n - p) \rangle. \tag{3.11}
\]

It follows from (3.9), (3.11) and Lemma 2.4 that
\[
\lim_{n \to \infty} \|x_n - p\|^2 = 0.
\]

This completes the proof of the theorem.

**Remark 3.3.** It is interesting to have strong convergence without assuming compactness condition, but it has still shortcomings as number of parameters is four.

The following questions remain open.

1. It is not clear whether our main result holds true in a real uniformly convex Banach space or not.
2. Similarly, it is not clear whether our main result holds true in a real uniformly smooth Banach space or not.
3. We were not sure whether it possible to generalize the theorem to strictly pseudocontractive (or pseudocontracrive mappings) or not.
4. It is not clear what conditions on control parameters are imposed to secure strong convergence of the iteration in (1.5).

**References**


