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Some Threshold Results on An Ammensal-Enemy Ecological Model with Variable Ammensal Coefficient

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Abstract

The paper consists the threshold results on a Mathematical model of an Ammensal-Enemy Ecological Model With Variable Ammensal Coefficient for carrying out the stability analysis by Gause's Law of competitive exclusion (1934). A pair of non-linear system of ordinary differential equations characterizes the model and the four equilibrium points are identified. Some threshold results are derived to establish the stability of the co-existent Equilibrium State.

AMS Classification: 92 D 25, 92 D 40

Keywords : Equilibrium states, Stability, Principle of Competitive exclusion. Threshold results

Introduction

In the field of Mathematical modeling apart from other branches of it Ecology occupies an important and prominent place. The word Ecology is derived from Greek. It means "The interdisciplinary scientific study of interactions between organisms and their environment". It also studies the ecosystems. Ecosystems give us the information of the network about the relations among the organisms at different scales. Ecology can refer to any form of bio-diversity. Hence ecologists are capable of conducting research from the smallest bacteria to the global atmospheric gases. Ecology is a modern discipline of science which came into existence in 19th century. It blossomed from natural science. Ecology is not synonymous to environment, environmentalism or environment science. The main scientific disciplines of physiology, evolution, genetics and behaviour are closely related to Ecology. In short we can say that Ecology is the study of living beings such as animals and plants in relation to their habits and habitats. It mainly deals with the evolutionary biology which explains us about how the living beings are regulated in nature.Mathematical modeling of ecosystems was initiated in 1925 by Lotka [16] and in 1931 by Volterra [20]. The general concepts of modeling have been presented in the treatises of Meyer [17], Paul Colinvaux [18],kushingJ.M[13], Kapur [12] and several authors.. The ecological symbiosis can be broadly classified as Prey-Predation, Competition, Mutualism, Commensalism, Ammensalism, and so on. N.C. Srinivas [19] studied the competitive ecosystems of two and three species with limited and unlimited resources. Lakshminarayan and Pattabhiramacharyulu [14,15] investigated Prey-predator Ecological models with a partial cover for the prey and alternate food for the predator. Recently, stability analysis of competitive species was carried out by Archana Reddy,N.Ch.Pattabhirama charyulu and Gandhi [11]. The present authors Acharyulu and Pattabhi Ramacharyulu [1-10] elicited some remarkable results “on the stability of an Ammensal -enemy species pair with various resources.

The paper is related to an analytical study on some threshold results of an Ammensal-Enemy Ecological Model with Variable Ammensal Coefficient. A pair of non-linear system of ordinary differential equations characterizes the model and the four equilibrium states are identified and some threshold results are derived to establish the stability of the co-existent Equilibrium State.

Notation Adopted:

$N_1(t)$: The population of the Ammensal species.

$N_2(t)$: The population of the enemy species

a_i : The natural growth rate of N_i, i = 1, 2

a_{ii} : The rate of decrease of N_i, due to insufficient resources of N_i, i = 1, 2

$K_i = \dfrac{a_i}{a_{ii}}$: Carrying capacities of N_i, i = 1, 2

α, β = co-efficients of Ammensalisms, $\beta \neq 0$.

The state variables N_1 and N_2 as well as the model parameters $a_1, a_2, a_{11}, a_{22}, K_1, K_2$ are assumed to be non-negative constants. Also α, β are coefficients of Ammensalisms. The interaction would be neutral when any one of them is equal to zero. The Ammensalisms is low or high according to β >0 or β <0.

Basic Balancing Equations of the Model

I. Equation for the growth rate of Ammensal species(N_1):

$$\frac{dN_1}{dt} = N_1[a_{11}(K_1 - N_1) - F(N_2)] \tag{1}$$

II. Equation for the growth rate of enemy species(N_2):

$$\frac{dN_2}{dt} = a_{22}N_2[K_2 - N_2] \tag{2}$$

In the equation (1) the function $F(N_2)$ is the characteristic of the Ammensalisms of N_1 with respect to the enemy N_2

$Lt\ F(N_2)$ with the condition that $F(N_2)$ is bounded for large N_2 .A reasonably simple choice of $F(N_2)$ is Monad type given by Kapur []:

$F(N_2) = \dfrac{\alpha N_2}{\beta + N_2}$ where $\alpha = \underset{N_2 \to \infty}{Lt} F(N_2), \alpha$ is a parametric characteristic of Ammensalism.Further β ($\neq 0$)is another parameter signifying the strength of the Ammensalisms. If β>0 then it is weak Ammensalism and if β<0 then it is strong Ammensalism.

Equilibrium States

The system under investigation has four equilibrium states.

i. Fully washed out state: $\overline{N_1} = 0;\ \overline{N_2} = 0$ (3)

The state in which the Ammensal species($\overline{N_1}$) is washed out while the enemy species($\overline{N_2}$) only survives: $\overline{N_1} = 0;\ \overline{N_2} = K_2$ (4)

The state in which the enemy species($\overline{N_2}$) is washed out while the Ammensal species($\overline{N_1}$) only survives: $\overline{N_1} = K_1;\ \overline{N_2} = 0$ (5)

The state in which both the species co-exist.

$$\overline{N_1} = \frac{1}{\alpha_{11}}\left(K_1\alpha_{11} - \frac{\alpha K_2}{\beta + K_2}\right) = e_1\ ;\quad \overline{N_2} = K_2 = e_2 \tag{6}$$

This state may also be called as the “normal steady state”. and this state exists only when $\left(K_1 a_{11} - \dfrac{\alpha K_2}{\beta + K_2}\right) > 0$

Threshold Diagrams

Gause’s Priciple of competitive exclusion: “Two species cannot indefinitely co-exist in the same locality if there have identical ecological requirements “

In consonance with the above principle, we intend to derive threshold diagrams according to Gause’s principle of competitive exclusion of the following cases where there are these possibilities to draw the direction fields in a phase plane.

CASE (A) : $\alpha < a_1$ (7)

CASE (B) : $\alpha > a_1$ (8)

It is divided in to three sub cases

Sub case i) $\beta > \dfrac{K_2(\alpha - a_1)}{a_1}$

Sub case ii) $\beta = \dfrac{K_2(\alpha - a_1)}{a_1}$

Sub case iii) $\beta > \dfrac{K_2(\alpha - a_1)}{a_1}$

CASE (A) : when $\alpha < a_1$

In view of Gause's Law of competitive exclusion for co-exist equilibrium state $\overline{N_1} = \frac{1}{\alpha_{11}}(K_1\alpha_{11} - \frac{\alpha K_2}{\beta + K_2}) = e_1$; $\overline{N_2} = K_2 = e_2$ $\overline{N_1} = \frac{1}{\alpha_{11}}(K_1\alpha_{11} - \frac{\alpha K_2}{\beta + K_2}) = e_1$; $\overline{N_2} = K_2 = e_2$ $\overline{N_1} = \frac{1}{\alpha_{11}}(K_1\alpha_{11} - \frac{\alpha K_2}{\beta + K_2}) = e_1$; $\overline{N_2} = K_2 = e_2$ then every solution of $N_1(t)$, $N_2(t)$ A approaches the equilibrium solution $N_1(t) = \overline{N_1}(\neq 0)$ and $N_2(t) = \overline{N_2}(\neq 0)$ as t approaches infinity.

i.e. if both species are identical and the microcosm can support both the species depending upon the initial conditions.

We divide the first quadrant into four regions in which $\dot{N}_1$ and $\dot{N}_2$ are having the fixed signs

Signs of $\dot{N}_1$ and $\dot{N}_2$ in the specific regions :

Region I : $N_1(t)$ and $N_2(t)$ flourish with time t.

Region II : $N_1(t)$ flourishes and $N_2(t)$ declines with time 't'.

Region III : $N_1(t)$ declines and $N_2(t)$ flourishes with time 't'.

Region IV : Both the species $N_1(t)$ and $N_2(t)$ decline with time 't 'and also both Ammensal - enemy species compete with each other but neither increase nor get extinct at time 't'.

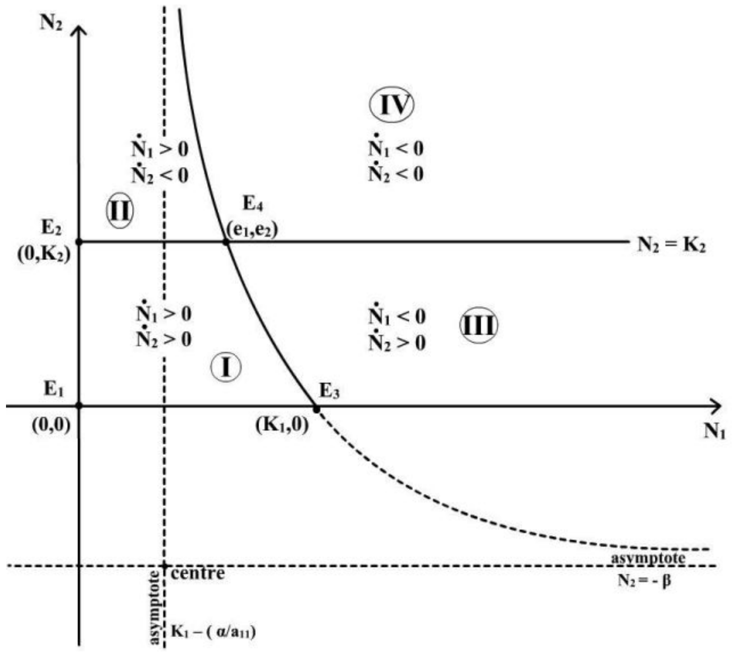

Figure 1

For deriving threshold diagrams to this case, we need the following Lemmas.

Lemma(I) : The solution which starts in region I at time $t = t_0$ of $N_1(t)$ and $N_2(t)$ will remain in this region for all future time $t \geq t_0$ and ultimately tends towards equilibrium solution $N_1(t) = \overline{N}_1$, $N_2(t) = \overline{N}_2$.

Lemma (II) : The solution which starts in region II at time $t = t_0$ of $N_1(t)$ and $N_2(t)$ will remain in this region for all future time $t \geq t_0$ and ultimately approaches the equilibrium solution $N_1(t) = \overline{N}_1$, $N_2(t) = \overline{N}_2$.

Lemma (III) : The solution which starts in region - III at time $t = t_0$ of $N_1(t)$ and $N_2(t)$ will remain in this region for all future time $t \geq t_0$ and ultimately reaches the equilibrium solution $N_1(t) = \overline{N}_1$, $N_2(t) = \overline{N}_2$.

Lemma (IV) : The solution of $N_1(t)$ and $N_2(t)$ which starts in the region IV for all future time $t \geq t_0$ and ultimately approaches the equilibrium solution $N_1(t) = \overline{N}_1$, $N_2(t) = \overline{N}_2$.

By the concept of Threshold diagrams and above Lemmas, we can conclude that every solution $N_1(t)$, $N_2(t)$ of (1) and (2) starts in region I, II, III and IV at time $t = t_0$ and remains there for all future time and finally they approach equilibrium solution $N_1(t) = \overline{N}_1$, $N_2(t) = \overline{N}_2$ as 't' approaches infinity.

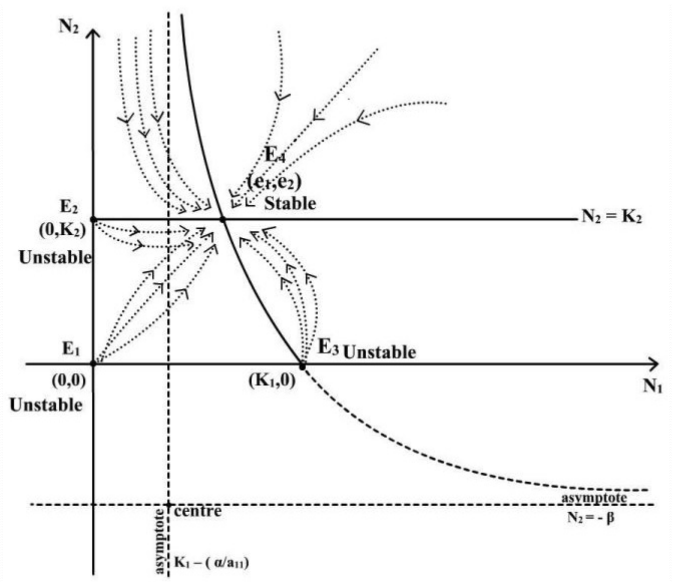

Figure 2

Conclusion

By observing all four regions as shown in the above figure, all the solutions which start in respective regions finally approach the equilibrium solution E_4 (e_1,e_2). It concludes that the state corresponding to the equilibrium point is STABLE.

Case(B) : if $\alpha > a_1$

Sub case(i) : when $\alpha > a_1$ and $\beta > \dfrac{K_2(\alpha - a_1)}{a_1}$

i.e $K_1 < \dfrac{\alpha}{a_{11}}$ and $\beta < \dfrac{K_2(\alpha - a_1)}{a_1} = \dfrac{K_2\alpha}{a_1} - K_2$

i.e $\dfrac{K_1\beta}{\dfrac{\alpha}{a_{11}} - K_1} < K_2$

Now $\beta < \dfrac{K_2\alpha}{a_1} - K_2$,

$$\beta + K_2 < \frac{K_2\alpha}{a_1} \Rightarrow \frac{1}{\beta + K_2} > \frac{a_1}{K_2\alpha}$$

$$\Rightarrow \frac{\alpha K_2}{\beta + K_2} > a_1$$

$$\therefore a_1 - \frac{\alpha K_2}{\beta + K_2} < 0$$

$i.e\,\frac{1}{a_{11}}[K_1 a_{11} - \frac{\alpha K_2}{\beta + K_2}] < 0$

which is a contradiction to $\overline{N_1} > 0$ *in* E_4

Thus this case does not exist

Hence the concept of threshold diagrams is not applicable to this case. (9)

Sub case(i) : when $\alpha > a_1$ and $\beta = \frac{K_2(\alpha - a_1)}{a_1}$

In this cases we have $K_1 < \frac{\alpha}{a_{11}}$ and $\beta = \frac{K_2(\alpha - a_1)}{a_1}$ $(i.e\ \frac{K_1\beta}{\frac{\alpha}{a_{11}} - K_1} = K_2)$

if $\alpha < a_1$ *and* $\beta = \frac{K_2(\alpha - a_1)}{a_1}$

$\Rightarrow \beta + K_2 = \frac{K_2(\alpha - a_1)}{a_1} + K_2$

$\Rightarrow \beta + K_2 = \frac{K_2\alpha - a_1 K_2 + K_2 a_1}{a_1} = \frac{K_2\alpha}{a_1}$

$\therefore \beta + K_2 = \frac{K_2\alpha}{a_1}$

X- coordinate of E_4 becomes

$\frac{1}{a_{11}}[K_1 a_{11} - \frac{\alpha K_2}{\beta + K_2}] = \frac{1}{a_{11}}[a_1 - \frac{\alpha K_2}{\frac{K_2\alpha}{a_1}}] = \frac{1}{a_{11}}[a_1 - a_1] = 0$

The equilibrium point (E_4) coincides with E_2 i.e $(0,K_2)$.

Hence it becomes unstable. (10)

Sub case (iii) : when $\alpha > a_1$ and $\beta > \frac{K_2(\alpha - a_1)}{a_1}$

CASE (3) : if $\alpha < a_1$ and $\beta > \frac{K_2(\alpha - a_1)}{a_1}$

if $\alpha < a_1$ & $\beta > \frac{K_2(\alpha - a_1)}{a_1}$

$\beta > \frac{K_2(\alpha - a_1)}{a_1} \Rightarrow \beta + K_2 > \frac{K_2(\alpha - a_1)}{a_1} + K_2 = \frac{K_2\alpha}{a_1}$

$i.e\ \beta + K_2 > \frac{K_2\alpha}{a_1} \Rightarrow a_1 > \frac{\alpha K_2}{\beta + K_2}$

X- coordinate of E_4 becomes $\frac{1}{a_{11}}[K_1 a_{11} - \frac{\alpha K_2}{\beta + K_2}] = \frac{1}{a_{11}}[a_{11} - \frac{\alpha K_2}{\beta + K_2}] > 0$

Thus $\overline{N_1} > 0$ occurs only when $\dfrac{K_1\beta}{\dfrac{\alpha}{a_{11}} - K_1} > K_2$

The threshold diagram can be illustrated as below. In this case we illustrate the diagram which emphasizes the signs of $\dot{N}_1$ *and* $\dot{N}_2$ with reference to the three regions by Gauss's principle of competitive exclusion shown in the following Figure 3.

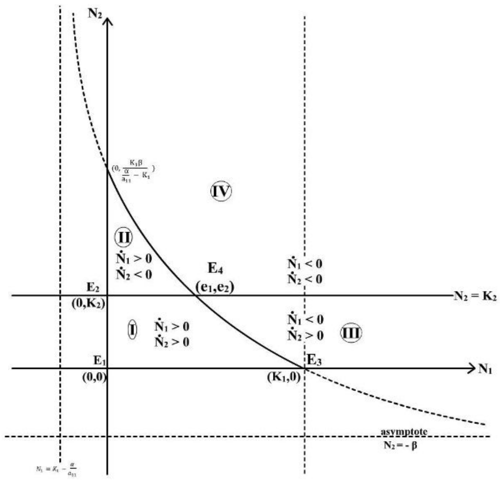

Figure 3

Here the threshold diagram is illustrated as in Fig. 4.

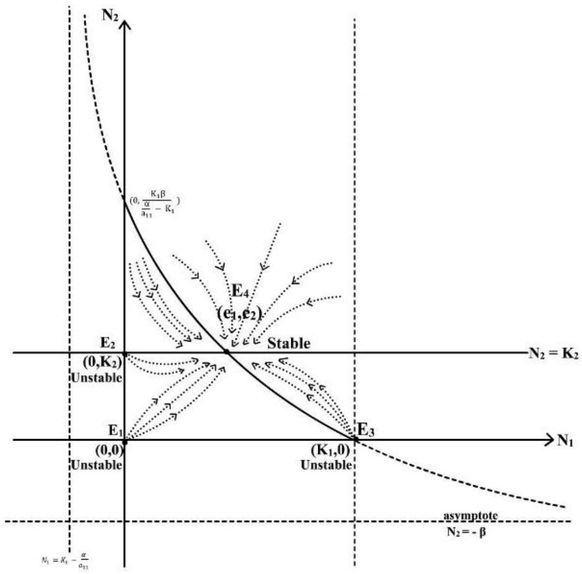

Figure 4

It is identified that all the solutions which start in respective regions finally reach the equilibrium solution E_4 (e_1, e_2). It concludes that the state E_4 corresponding to the equilibrium point(e_1,e_2) is STABLE.

Conclusion

The stability analysis is carried out by Gause's Law of competitive exclusion (1934) on an Ammensal-Enemy Ecological Model with Variable Ammensal Coefficient in the case of normal steady state.

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