A Subclass of Analytic Functions

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Abstract

Let \( S(A, B) \) denote the class of functions \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \), regular in the unit disc \( E = \{ z : |z| < 1 \} \) and satisfying the condition

\[
\frac{f(z)}{g(z)} \prec 1 + \frac{A z}{1 + B z},
\]

where \( A \geq -1 \) and \( B \leq 1 \), \( z \in E \), where \( g(z) \) is starlike in \( E \). In this paper, we obtain the coefficient estimates, distortion theorem, argument theorem and radius of starlikeness for the class \( S(A, B) \).

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Introduction

Let \( U \) be the class of bounded functions

\[
w(z) = \sum_{k=1}^{\infty} c_k z^k
\]

which are regular in the unit disc \( E = \{ z : |z| < 1 \} \) and satisfying the conditions

\[
w(0) = 0 \quad \text{and} \quad |w(z)| < 1, \quad z \in E.
\]

Let \( S' \) denote the class of functions

\[
g(z) = z + \sum_{k=2}^{\infty} b_k z^k
\]

regular and starlike in \( E \).
Let \( S_1(\alpha, \beta) \) be the class of functions \( f(z) \) which are regular in the unit disc \( E \) and satisfy the condition
\[
\left| \frac{f(z)}{g(z)} - 1 \right| < \beta \left| \frac{\lambda f(z)}{g(z)} + 1 \right|, \quad z \in E, \ 0 \leq \lambda \leq 1, \ 0 < \beta \leq 1,
\]
where \( g(z) \) is regular and starlike of order \( \alpha \ (0 \leq \alpha < 1) \).

This class was studied by Goel and Sohi [3].

Let \( S(A, B) \) denote the class of functions
\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k\tag{1.3}
\]
regular in \( E \) and satisfying the conditions
\[
\frac{f(z)}{g(z)} < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in E, \ g \in S^*. \]

Obviously \( S(A, B) \) is a subclass of close-to-star functions introduced by Reade [4].

Also \( S(1, -1) \equiv S_1(\alpha, \beta)(\lambda=1, \alpha=0, \beta=1) \).

By definition of subordination it follows that \( f(z) \in S(A, B) \), if and only if \( f(z) \) can be expressed in the form
\[
\frac{f(z)}{g(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad w(z) \in U, \ -1 \leq B < A \leq 1, \ z \in E. \tag{1.4}
\]

To avoid repetition, we lay down once for all that \( -1 \leq B < A \leq 1, \ z \in E \).

The purpose of this paper is to study the class \( S(A, B) \) and obtain coefficient estimates, distortion theorem, argument theorem and radius of starlikeness.

**Preliminary Lemmas**

**Lemma 2.1.** Let
\[
\frac{f(z)}{g(z)} = P(z) = 1 + \sum_{k=2}^{\infty} p_k z^k, \tag{2.1}
\]
then
\[
|p_n| \leq (A - B), \quad n \geq 1. \tag{2.2}
\]

This lemma is due to Goel and Mehrok [1].

**Lemma 2.2.** If \( w(z) \in U \), then for \( |z| = r < 1 \),
\[
|zw'(z) - w'(z)| \leq \frac{r^2 - |w(z)|^2}{1 - r^2}. \tag{2.3}
\]

Singh and Goel proved this result in [5].

**Lemma 2.3.** Let \( p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, \ w(z) \in U \), then for \( |z| = r < 1 \),
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\[
\text{Re} \left[ Ap(z) + \frac{B}{p(z)} \right] + r^2 \frac{|Ap(z) - B|^2 - |1 - p(z)|^2}{(1 - r^2)|p(z)|}
\]

\[
\leq \left\{ \begin{array}{ll}
AB(A + B) r^2 - 4ABr + (A + B) & , R_i \leq R_0, \\
 \frac{2}{(1 - r^2)} \left[ (1 - ABr^2) - (1 - A)(1 - B)(1 + Ar^2)(1 + Br^2) \right]^{1/2} & , R_i \geq R_0, A \neq 1,
\end{array} \right.
\]

where \( R_i = \frac{1 - Br}{1 - Ar} \) and \( R_0^3 = \frac{(1 - B)(1 + Br^2)}{(1 - A)(1 + Ar^2)} \).

The bounds are sharp.

Goel and Mehrok [2] established this result.

**Coefficient Estimates**

**Theorem.** 3.1. If \( f(z) \in S(A, B) \), then

\[
|a_n| \leq n \left[ 1 + \frac{(n-1)(A-B)}{2} \right], \quad n \geq 2.
\]

The bounds are sharp.

**Proof.** Using (1.2) and (1.3) in (2.1), we get

\[
z + \sum_{k=2}^{\infty} a_k z^k = \left( z + \sum_{k=2}^{\infty} b_k z^k \right) \left( 1 + \sum_{k=1}^{\infty} p_k z^k \right).
\]

On equating the coefficients of \( z^n \) in (3.2), we have

\[
a_n = b_n + p_1 b_{n-1} + p_2 b_{n-2} + \ldots + p_{n-1}.
\]

Using (2.2), (3.3) yields

\[
|a_n| \leq |b_n| + (A - B) \left[ |b_{n-1}| + |b_{n-2}| + \ldots + |b_2| + 1 \right].
\]

Also it is well known that \( |b_n| \leq n \), \( n \geq 2 \). Hence

\[
|a_n| \leq n \left[ 1 + \frac{(n-1)(A-B)}{2} \right].
\]

For \( n = 2 \), equality signs in (3.1) hold for the functions \( f_n(z) \) defined by

\[
f_n(z) = \frac{z}{1 - \delta z} \left( 1 + A\delta z^{n-1} \right), \quad |\delta| = 1, \ |\delta_2| = 1.
\]

On putting \( A = 1, B = -1 \), we get the following result due to Reade [4].

**Corollary.** Let \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \) is close-to-star function, then

\[
|a_n| \leq n^2, \quad n \geq 2.
\]
Distortion Theorem

Theorem 4.1. If \( f(z) \in S(A, B) \), then for \( |z| = r \), \( 0 < r < 1 \), we have

\[
\frac{r(1-Ar)}{(1-Br)(1+r)^2} \leq |f(z)| \leq \frac{r(1+Ar)}{(1+Br)(1-r)^2}.
\]

Estimate is sharp.

Proof. From (1.4), we have

\[
|f(z)| = |g(z)| \frac{|1+Aw(z)|}{1+Bw(z)}, \quad w(z) \in U. \tag{4.2}
\]

It is easy to show that

\[
\frac{1-Ar}{1-Br} \leq \frac{|1+Aw(z)|}{1+Bw(z)} \leq \frac{1+Ar}{1+Br}. \tag{4.3}
\]

Since \( g(z) \) is starlike, it follows that

\[
\frac{r}{(1+r)^2} \leq |g(z)| \leq \frac{r}{(1-r)^2}. \tag{4.4}
\]

Using (4.3) and (4.4) in (4.2), we obtain (4.1). For \( n = 2 \), the function \( f_n(z) \) defined by (3.4), gives sharp estimates.

For \( A = 1, B = -1 \), we have the following:

Corollary. Let \( f(z) \in S(1,-1) \), then for \( |z| = r \), \( 0 < r < 1 \),

\[
\frac{r(1-r)}{(1+r)^3} \leq |f(z)| \leq \frac{r(1+r)}{(1-r)^3}.
\]

This result was proved by Goel and Sohi [3].

Argument Theorem

Theorem 5.1. If \( f(z) \in S(A,B) \), then

\[
\left| \arg \frac{f(z)}{z} \right| \leq 2 \sin^{-1} r + \sin^{-1} \frac{(A-B)r}{1-ABr^2}. \tag{5.1}
\]

The result is sharp.

Proof. It is easy to show that, the transformation

\[
\frac{f(z)}{g(z)} = \frac{1+Aw(z)}{1+Bw(z)}
\]

maps \( |w(z)| \leq r \) onto the circle.
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\[
\left| \frac{f(z)}{g(z)} \right| \leq \left| \frac{1 - ABr^2}{1 - B^2r^2} \right| \leq \frac{(A - B)r}{1 - ABr^2}, \quad |z| = r.
\]

Therefore
\[
\left| \frac{f(z)}{g(z)} \right| \leq \sin^{-1} \left( \frac{(A - B)r}{1 - ABr^2} \right).
\]

Also
\[
\left| \frac{g(z)}{z} \right| \leq 2\sin^{-1} r.
\]

Since
\[
\left| \frac{f(z)}{z} \right| \leq \left| \frac{f(z)}{g(z)} \right| + \left| \frac{g(z)}{z} \right|.
\]

Using (5.2) and (5.3) in (5.4) follows. The result (5.1) is sharp for the function \( f_n(z) \) \((n = 2)\) defined by (3.4), where
\[
\delta_z = \frac{r}{z} \left[ \frac{-(A + B)r + i \left( (1 - A^2 r^2)(1 - B^2 r^2) \right)^{1/2}}{1 + ABr^2} \right].
\]

On putting \( A = 1, B = -1 \), we obtain the following result due to Goel and Sohi [3].

**Corollary.** Let \( f(z) \in S(1, -1) \), then
\[
\left| \frac{f(z)}{z} \right| \leq 2\sin^{-1} r + \sin^{-1} \frac{2r}{1 + r^2}.
\]

**Radius of Starlikeness**

**Theorem.** 6.1. Let \( f(z) \in S(A, B) \), then

For \( A_b \leq A \leq 1 \), \( f(z) \) is starlike in \( |z| < r_b \), where \( r_b \) is the smallest positive root of
\[
ABr^3 - B(2 + A)r^2 + (1 + 2A)r - 1 = 0;
\]

For \(-1 < A \leq A_b \), \( f(z) \) is starlike in \( |z| < r_1 \), where \( r_1 \) is the smallest positive root of
\[
B(1 - A)r^4 - 2B(1 - A)r^3 + (1 - 2(A - B) - ABr^2 - 2(1 - A)r + (1 - A) = 0;
\]

\[
A_b = \left( \frac{3 - \sqrt{5}}{2} \right).
\]

Results are sharp.

**Proof.** Differentiating (1.4) logarithmically, it yields
\[
\frac{zf'(z)}{f(z)} = \frac{zg'(z) + (A - B)zw'(z)}{g(z)} \quad (6.3)
\]

Taking the real parts on both sides of (6.3) and using lemma 2.2, we get

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) \geq \Re \left( \frac{zg'(z)}{g(z)} \right) + (A - B) \left( \frac{zw'(z)}{g(z)} \right)
\]

\[
+ (A - B) \left[ \Re \left( \frac{w(z)}{(1 + Aw(z))(1 + Bw(z))} \right) - \frac{r^2 - |w(z)|^2}{(1 - r^2) \left| (1 + Aw(z))(1 + Bw(z)) \right|} \right]. \quad (6.4)
\]

Put \( p(z) = \frac{1 + Bw(z)}{1 + Aw(z)} \), \( w(z) \in U \).

Then from (6.4), we have

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) \geq \Re \left( \frac{zg'(z)}{g(z)} \right) + \frac{(A + B)}{(A - B)} \left( \frac{zw'(z)}{g(z)} \right)
\]

\[
- \frac{1}{(A - B)} \left[ \Re \left( Ap(z) + \frac{B}{p(z)} \right) + \frac{r^2 A p(z) - B^2 [1 - p(z)]^2}{(1 - r^2) \left| p(z) \right|^2} \right]. \quad (6.5)
\]

Since \( g(z) \) is starlike, therefore we have

\[
\Re \left( \frac{zg'(z)}{g(z)} \right) \geq \frac{1 - r}{1 + r}. \quad (6.6)
\]

(6.5) together with (6.6) and lemma 2.3, yields

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) \geq \frac{1 - (1 + 2A) r + B(2 + A)^2 - 3ABr^3}{(1 + r)(1 - Ar)(1 - Br)}, R_i \leq R_{0},
\]

\[
\geq \frac{-2[(1 - A) + (A - B) r + B(1 - A) r^2]}{(1 - A)(1 - B)(1 + Ar^2)(1 + Br^2)}^{1/2}, R_i \geq R_{0}, A \neq 1. \quad (6.7)
\]

On equating the right hand sides of (6.7) to zero, we get (6.1) and (6.2).

The equation \( R_i = R_0 \) yields

\[
ABr^3 - 2ABr^3 + (2A + 2B - AB - 1) r^2 - 2r + 1 = 0. \quad (6.8)
\]

Elimination of \( r \) between (6.1) and (6.8) leads to

\[
(1 + B)(BA^3 - 2BA^2 + 2A - 1) = 0. \quad (6.9)
\]

If \( 1 + B \neq 0 \), we have
\[ B = \frac{(2A-1)}{A^2(2-A)}, \quad A \neq 1. \]

Then \( B < A \) implies that \( 0 < (1-A)^3(1+A) \) which holds.

Also \( B = \frac{(2A-1)}{A^2(2-A)} > -1 \) implies \( A < \frac{3-\sqrt{5}}{2} < 1. \)

For \( B = -1 \), elimination of \( r \) between (6.1) and (6.8) gives \( A^2 - 3A + 1 = 0. \)

Therefore \( A = \frac{3-\sqrt{5}}{2} = A_0 \), say.

By taking \( A = 1, B = -1 \), we have the following result.

**Corollary.** Let \( f(z) \in S(1,-1) \), then \( f(z) \) is starlike in \( \{|z| < r_2\} \) where \( r_2 \) is the smallest positive root of the equation

\[ r^3 - 3r^2 - 3r + 1 = 0. \]

This result is due to Goel and Sohi [3].

**References**


