Lie Symmetries for a 2D Non Linear Heat Equation with Source

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Abstract

The paper investigates a general form of the 2D non linear heat equation with source and it points out possible cases where type symmetries lie and similarity solution can appear.

Keywords: Lie symmetry approach; Non-linear heat Equations.

Introduction

The non linear heat equation has been extensively studied in 1D context [1-3]. We will consider the 2D non linear heat equation with a source

\[ u_t = (k(u)u_x)_x + q(u) \]  

(1)

In the above equation if \( q(u) \) and \( y \) is replaced by \( x \) then a true non linear heat equation is obtained

\[ u_t = (k(u)u_x)_x \]  

(2)

The classical symmetry method was firstly applied to (2) in [8] and results were extended to the non linear heat equation with convention term in [9], further \( k(u) = 0 \) in (2) yields the linear heat equation. Adding a source term in equation (2) gives

\[ u_t = (k(u)u_x)_x + q(u) \]  

(3)

the lie symmetry of equation (3) was completely described in [4], and the conditional symmetry of nonlinear heat equation (3) were studied in [5-6].

A more general form of equation (2) \( u_t = (k(u)u_x)_y \) was studied in [7]. In [7] if
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$k(u) = u^{-1}$ then equation becomes $u_t = (u^{-1}u_x)_x$, which on simplifying gives

$u_t = -u_x u_x u^{-2} + u_{xx} u^{-1}$

the equation obtained is the 2D ricci flow an equation with large implications in gravity theory.

The General Prolongation formula

Let $v = \sum_{i=1}^{p} \xi^i (x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_{\alpha}(x,u) \frac{\partial}{\partial u^\alpha}$

be a vector field defined on an open subset $M \subset X \times U$ where $X$ is the space of independent variables, and $U$ is the space of dependent variables, $p$ is the number of independent variables and $q$ is the number of dependent variables for the system. Then $n^{th}$-prolongation of $v$ is defined on the corresponding jet space $M^{(n)} \subset X \times U^{(n)}$ where $X$ is the space of independent variables, $U^{(n)}$ is the space of the dependent variables and the derivative of the dependent variables up-to $n$ (order of differential equation) by

$$pr^{(n)}v = v + \sum_{\alpha=1}^{q} \sum_{J} \phi_{\alpha}^{J}(x,u^{(n)}) \frac{\partial}{\partial u_{j}^{\alpha}}$$

The second summation being over all unordered multi-indices $J = (j_1, j_2, \ldots, j_k)$ with $1 \leq j_k \leq p$, $1 \leq k \leq n$. The coefficient function $\phi_{\alpha}^{J}$ of $pr^{(n)}v$ are given by the formula [10]

$$\phi_{\alpha}^{J}(x,u^{(n)}) = D_J \left( \phi_{\alpha} - \sum_{i=1}^{n} \xi^i u^\alpha_i \right) + \sum_{i=1}^{n} \xi^i u_{j,i}^\alpha$$

where $u_i^\alpha = \left( \frac{\partial u^\alpha}{\partial x^i} \right)$ and $u_{j,i}^\alpha = \left( \frac{\partial u_{j}^\alpha}{\partial x^i} \right)$.

Theorem

Suppose $\Delta_d(x, u^{(n)}) = 0$, for $d = 1, \ldots, l$ is a system of differential equations of maximal rank defined over $M \subset X \times U$. If $G$ is a local group of transformations acting on $M$, and

$$pr^{(n)}v \left[ \Delta_d(x, u^{(n)}) \right] = 0$$

(for $d = 1, \ldots, l$, whenever $\Delta(x, u^{(n)}) = 0$)

for every infinitesimal generator $v$ of $G$, then $G$ is a symmetry group of the system.

General approach

The lie symmetries for a PDE equation are the classical ones which keep the equation
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invariant under a lie group of local infinitesimal transformations. In this case we investigate the lie symmetries of (1)

\[ u_t = (k(u)u_x)_x + q(u) \]

equation (1) on simplifying becomes

\[ u_t = k(u)u_{xx} + k'(u)u_{xy} + q(u) \quad (8) \]

let us consider a one –parameter group of point like transformations acting on the space of independent variables \((x,y,t)\) and that of the dependent variable of equation (8) with the associated infinitesimal generator given by

\[ \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \phi \frac{\partial}{\partial u} + \tau \frac{\partial}{\partial t} \]

Where \( v(x, y, t, u), \xi(x, y, t, u), \eta(x, y, t, u), \phi(x, y, t, u), \tau(x, y, t, u) \)
The invariance condition (7) for the second order PDE equation (8) is

\[ pr^{(2)}v (u_t - k(u)u_{xy} - k'(u)u_{xy} - q(u)) = 0 \quad (10) \]

Where,

\[ pr^{(2)}v = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \phi \frac{\partial}{\partial u} + \tau \frac{\partial}{\partial t} + \phi ' \frac{\partial}{\partial u_x} + \phi '' \frac{\partial}{\partial u_y} + \phi ' \frac{\partial}{\partial u_t} \]

The condition (10) becomes

\[ \phi' = \phi k'(u)u_{xy} + \phi'' k(u) + \phi u_x u_{xy} k'(u) + \phi' k'(u)u_y + \phi' k'(u)u_x + \phi q(u) \quad (12) \]

Substituting the values of \( \phi', \phi'', \phi''', \phi''' \) using the formula (6), value of \( u_t \) from equation (8) and equating the coefficients of the terms in the first and higher order partial derivatives of \( u \), the determining equations for symmetry group of the equation (1) are found as follows

\[ \phi_x + (\phi_y - \tau)q(\omega) = \phi q'(\omega) + k(\omega)\phi_x \quad (13) \]

\[ -\xi_x - k(\omega)\phi_{xx} + k'(\omega)\phi_x \quad (14) \]

\[ -\eta_t = k(\omega)\phi_{xt} + k'(\omega)\phi_x \quad (15) \]

\[ k(\omega)(\xi_x + \eta_x + \tau) = \phi k'(\omega) \quad (16) \]

\[ k'(\omega)(\phi_x - \xi_x - \eta_x + \tau) + \phi k''(\omega) + k(\omega)\phi_{xt} = 0 \quad (17) \]
It is clear that, particular choices of \( k(u) \) and \( q(u) \), concrete expressions could be found for the associated Lie symmetry operators. In the next section we do a study on possible choices and we find the corresponding symmetries.

**Case 1 ;**
When \( k(u) = u^n, q(u) = u^n, n \neq a + 1 \)

In this case the 2D non-linear heat equation with source, (1), becomes
\[
u_t = (u^n u_x)_x + u^n \tag{18}
\]

For this chosen form of \( k(u) \) and \( q(u) \) the determining equations (13 – 17) of the classical symmetries associated with (1) are the following
\[
\phi_i + (\phi_u - \tau_\nu)u^n = \phi nu^{n+1} + u^n \phi_{u\nu}
\tag{19}
\]
\[
-\xi = u^n \phi_{uu} + au^{n-1} \phi
\tag{20}
\]
\[
-\eta = u^n \phi_{uu} + au^{n-1} \phi
\tag{21}
\]
\[
u^n (\xi_{uu} + \eta_{uu} + \tau_{\nu}) = au^{n-1} \phi
\tag{22}
\]
\[
u^n (\phi_u - \xi_{uu} - \eta_{uu} + \tau_{\nu}) + a(a - 1)\phi u^{n-2} + u^n \phi_{uuu} = 0
\tag{23}
\]

By solving the above system we get
\[
\xi = c_1 x + c_2, \quad \eta = c_3 y + c_4, \quad \tau = \frac{(n-1)}{(n-a-1)}c_4 t + \frac{(n-1)}{(n-a-1)}c_5 t + c_5
\]
\[
\phi = \left( -\frac{c_1}{(n-a-1)} - \frac{c_3}{(n-a-1)} \right) u
\]

The lie symmetry generator for (18) is
\[
v = (c_1 x + c_2) \partial_x + (c_3 y + c_4) \partial_y + \left( \frac{(n-1)}{(n-a-1)}c_4 t + \frac{(n-1)}{(n-a-1)}c_5 t + c_5 \right) \partial_t +
\]
\[
\left( \frac{c_1}{(n-a-1)} - \frac{c_3}{(n-a-1)} \right) u \partial_u
\]

The above expression suggests the existence of five independent lie symmetry
operators. They are

\[ v_1 = x \partial_x + \frac{(n-1)}{(n-a-1)} t \partial_t - \frac{u}{(n-a-1)} \partial_u, \quad v_2 = \partial_t, \]

\[ v_3 = x \partial_x + \frac{(n-1)}{(n-a-1)} t \partial_t - \frac{u}{(n-a-1)} \partial_u, \quad v_4 = \partial_x, \quad v_5 = \partial_t. \]

The operators \( v_2, v_4, v_5 \) generate the invariance of (18) under a group of infinitesimal space and time translations, respectively, the remaining generators \( v_1 \) and \( v_3 \) imply its invariance under scaling transformations.

When lie algebra of the above operators is calculated, the only non vanishing relations are obtained as \([v_1, v_2]=-v_2, [v_1, v_5]=-(n-1/n-a-1) v_5, [v_3, v_4]=- v_4,\) and \([v_3, v_5]=-(n-1/n-a-1) v_5\)

The one-parameter groups \( G_i \) (\( i=1,2,3,4,5 \)) generated by the \( v_i \) are given by using \( \exp(\varepsilon v_i)(x,y,t,u) \) as follows

\[ G_1(xe^{\varepsilon}, y, te^{(n-a-1)}, ue^{(n-a-1)}), \quad G_2(x+\varepsilon, y, t, u), \quad G_3(x, ye^{\varepsilon}, te^{(n-1)}, ue^{(n-a-1)}) \]

\[ G_4(x, y+\varepsilon, t, u), \quad G_5(x, y, t+\varepsilon, u) \]

Since each group \( G_i \) is a symmetry group.

The solution of non linear heat equation (18) corresponding to its different symmetry groups \( G_i \) (\( i=1,2,3,4,5 \)) are obtained by using \( \tilde{u} = g \cdot u = g \cdot f(x,y,t) \) as follows

\[ u^{(1)} = e^{\varepsilon} f(xe^{-\varepsilon}, y, te^{(1-n)}(n-a-1)), \quad u^{(2)} = f(x-\varepsilon, y, t), \quad u^{(3)} = e^{\varepsilon} f(x, ye^{-\varepsilon}, te^{(1-n)}(n-a-1)) \]

\[ u^{(4)} = f(x, y-\varepsilon, t), \quad u^{(5)} = f(x, y, t-\varepsilon) \]

where \( u = f(x, y, t) \) is the solution of non linear heat equation (18).

Case 2:

When \( k(u) = e^u, q(u) = e^{bu} \), \( b \neq 1 \)

In this case the 2D non linear heat equation with source (1), becomes

\[ u_t = (e^u u_x)_x + e^{bu} \quad (24) \]

For this chosen form of \( k(u) \) and \( q(u) \) the determining equations (13 – 17) of the classical symmetries associated with (1) are the following
\[ \phi_t + (\phi_x - \tau) e^{\phi u} = \phi b e^{\phi u} + e^\phi \phi_{xy} \] (25)

\[-\xi_t = e^\phi \phi_{ux} + e^\phi \phi_x \] (26)

\[-\eta_t = e^\phi \phi_{uy} + ae^{\phi u} \phi_x \] (27)

\[e^\phi (\xi_t + \eta_t + \tau_t) = e^\phi \phi \] (28)

\[e^\phi (\phi_u - \xi_t - \eta_t + \tau_t) + \phi e^u + e^\phi \phi_{uu} = 0 \] (29)

By solving the above system we get
\[ \xi = c_1 x + c_2, \quad \eta = c_3 y + c_4, \quad \tau = \frac{b}{(b-1)} c_1 t + \frac{b}{(b-1)} c_3 t + c_5 \]

\[ \phi = \left( \frac{-c_1}{(b-1)} - \frac{c_3}{(b-1)} \right) \]

The lie symmetry generator for (24) is
\[ v = (c_1 x + c_2) \partial_x + (c_3 y + c_4) \partial_y + \left( \frac{b}{(b-1)} c_1 t + \frac{b}{(b-1)} c_3 t + c_5 \right) \partial_t + \left( \frac{-c_1}{(b-1)} - \frac{c_3}{(b-1)} \right) \partial_u \]

The above expression suggests the existence of five independent lie symmetry operators. They are
\[ v_1 = x \partial_x + \frac{b}{(b-1)} t \partial_t - \frac{1}{(b-1)} \partial_u, \quad v_2 = \partial_x, \]

\[ v_3 = x \partial_x + \frac{b}{(b-1)} t \partial_t - \frac{1}{(b-1)} \partial_u, \quad v_4 = \partial_y, \quad v_5 = \partial_t \]

The operators \( v_2, v_4, v_5 \) generate the invariance of (24) under a group of infinitesimal space and time translations, respectively.

When lie algebra of the above operators is calculated then the only non-vanishing terms are \([ v_1, v_2] = -v_2, [ v_1, v_3] = -(b/b-1) v_5, [ v_3, v_4] = -v_4, [ v_3, v_5] = -(b/b-1) v_5 \)

The one-parameter groups \( G_i (i = 1,2,3,4,5) \) generated by the \( v_i \) are given by using
exp(εv)(x,y,t,u) as follows
\[ G_1(xe^\varepsilon, y, te^{\varepsilon/(b-1)}, u - \varepsilon/(b-1)), \quad G_2(x + \varepsilon, y, t, u), \quad G_3(x, ye^\varepsilon, te^{\varepsilon/(b-1)}, u - \varepsilon/(b-1)) \]
\[ G_4(x, y + \varepsilon, t, u), \quad G_5(x, y, t + \varepsilon, u) \]

Since each group \( G_i \) is a symmetry group.

The solution of non linear heat equation (24) corresponding to its different symmetry groups \( G_i (i = 1, 2, 3, 4, 5) \) are obtained by using \( \tilde{u} = g \cdot u = g \cdot f(x,y,t) \) as follows
\[ u^{(1)} = -\frac{-\varepsilon}{(b-1)} + f(xe^{-\varepsilon}, y, te^{-\varepsilon/(b-1)}), \quad u^{(2)} = f(x-\varepsilon, y, t), \quad u^{(3)} = -\frac{-\varepsilon}{(b-1)} + f(x, ye^{-\varepsilon}, te^{-\varepsilon/(b-1)}) \]
\[ u^{(4)} = f(x, y - \varepsilon, t), \quad u^{(5)} = f(x, y, t - \varepsilon) \]

where \( u = f(x, y, t) \) is the solution of non linear heat equation (24).

**Conclusion**

We analyzed the general form of 2D non linear heat equation with source which has a wide range of application in physics, engineering, chemistry, biology etc.

The general results have been applied for the two particular cases when \( k(u) = u^n, \quad q(u) = e^{nu} \) and \( k(u) = e^u, \quad q(u) = e^{bu} \).

The symmetry group provides a means of classifying different symmetry classes of solutions, where two solutions are deemed to be equivalent if one can be transformed into the other by some group element. In our investigation for the symmetry group \( G_1 \) and \( G_3 \) reflects the scaling of the equation, we can add solution and multiply them by constants. The group \( G_2, G_4 \) and \( G_5 \) are space translation symmetry groups of the equation.

An important extension of our present analysis seem to be of interest, the problem of non classical symmetries of (1), which will be tackled in a forthcoming paper.

**References**


