Global Journal of Mathematical Sciences: Theory and Practical.
ISSN 0974-3200 Volume 4, Number 1 (2012), pp. 49-55

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System of Intuitionistic Fuzzy Relational Equations

A.R. Meenakshi and T. Gandhimathi

Department of Mathematics, Karpagam University, India
E-mail: arm_meenakshi@yahoo.co.in, gandhimahes@rediffmail.com

Abstract

In this paper, we represent an intuitionistic fuzzy matrix as the Cartesian product representation of its membership and non-membership matrices. By using this representation, we introduce the concept of regularity for block IFMs and the consistency of intuitionistic fuzzy relational equations are discussed.

Keywords: Block fuzzy matrix, Block intuitionistic fuzzy matrix, Schur complement.

Introduction

We deal with fuzzy matrices that is, matrices over the fuzzy algebra F^M and F^N with support [0,1] and fuzzy operations {+,.} defined as a + b = max {a, b}, a.b = min{a,b} for all $a,b \in F^M$ and a + b = min {a, b}, a.b = max{a, b} for all $a, b \in F^N$. Let F^M_{mxn} be the set of all mxn Fuzzy matrices over F. A matrix $A \in F^M_{mxn}$ is said to be regular if there exists $X \in F^M_{nxm}$ such that AXA = A, X is called a generalized inverse (g-inverse) of A. In [2], Kim and Roush have developed the theory of fuzzy matrices, under max min composition analogous to that of Boolean matrices. Cho [1] has discussed the consistency of fuzzy matrix equations, if A is regular with a g-inverse X, then b.X is a solution of x A=b. Further every invertible matrix is regular. For more details on fuzzy matrices one may refer [4].Regularity of block fuzzy matrices and the consistency of a fuzzy relational equation with coefficient matrix is a block fuzzy matrix are discussed in[3]. The concept of intuitionistic fuzzy matrices (IFMs) as a generalization of fuzzy matrix was studied and developed by Madhumangal Pal et.al.[6].In our earlier work, we have studied on regularity of IFM[5].

In this paper, we discuss the consistency of Intuitionistic fuzzy relational equations as a generalization of fuzzy relational equations discussed in [4].In section 2,we present the basic definitions and required results on IFMs. In section 3,by introducing the concept of schur complements for a block IFM and discussed the consistency of intuitionistic fuzzy relational equations of the form xM=b, where M is a block intuitionistic fuzzy matrix and b is an intuitionistic fuzzy vector.

Preliminaries

Let $(IF)_{mxn}$ be the set of all intuitionistic fuzzy matrices of order mxn. Let $(IF)_{mxn}$ be the set of all intuitionistic fuzzy matrices of order mxn. First we shall represent A $\in (IF)_{mxn}$ as Cartesian product of fuzzy matrices. The Cartesian product of any two matrices A=$(a_{ij})_{mxn}$ and B=$(b_{ij})_{mxn}$, denoted as $\langle A,B\rangle$ is defined as the matrix whose ijth entry is the ordered pair $\langle A,B\rangle$= $(\langle a_{ij}, b_{ij}\rangle)$. For $A=(a_{ij})_{mxn}$ = $(\langle a_{ij\mu}, a_{ij\nu}\rangle)$ $\in (IF)_{mxn}$. We define $A_\mu = (a_{ij\mu}) \in$ F^{M}_{mxn} as the membership part of A and $A_\nu = (a_{ij\nu}) \in$ F^{N}_{mxn} as the non membership part of A. Thus A is the Cartesian product of A_μ and A_ν written as $A=\langle A_\mu, A_\nu\rangle$ with $A_\mu \in F^{M}_{mxn}$, $A_\nu \in F^{N}_{mxn}$.

Here we shall follow the matrix operations on intuitionistic fuzzy matrices as defined in our earlier work [5].

For $A,B \in (IF)_{mxn}$, if $A=\langle A_\mu, A_\nu\rangle$ and $B=\langle B_\mu, B_\nu\rangle$, then

(2.1) $$A+B=\langle A_\mu + B_\mu, A_\nu + B_\nu\rangle$$

For $A \in (IF)_{mxp}$, $B \in (IF)_{pxn}$ if $A=\langle A_\mu, A_\nu\rangle$ and $B=\langle B_\mu, B_\nu\rangle$, then

(2.2) $$AB=\langle A_\mu . B_\mu,\ A_\nu . B_\nu\rangle$$

$A_\mu . B_\mu$ is the max min product in F^{M}_{mxn}, $A_\nu . B_\nu$ is the min max product in F^{N}_{mxn}.

For $A \in (IF)_{mxn}$, R(A) (C(A)) be the space generated by the rows (columns) of A. Let us define the order relation on $(IF)_{mxn}$ as,

(2.3) $$A \le B \Leftrightarrow A_\mu \le B_\mu \text{ and } A_\nu \ge B_\nu \Leftrightarrow A+B=B.$$

In the sequel, we shall make use of the following results proved in our earlier work [5].

Lemma 2.1[5]: For $A,B \in (IF)_{mxn}$, R(B)$\subseteq$R(A) $\Leftrightarrow$ B=YA for some Y$\in (IF)_n$, C(B) $\subseteq$C(A) $\Leftrightarrow$ B = AX for some X$\in$ $(IF)_n$.

Lemma 2.2[5]: Let $A \in (IF)_{mxn}$ be of the form $A=\langle A_\mu, A_\nu\rangle$. Then A is regular $\Leftrightarrow$ A_μ is regular in F^{M}_{mxn} under max min composition and A_ν is regular in F^{N}_{mxn} under min max composition.

Lemma 2.3[5]: If $A \in (IF)_{mxn}$ is of the form $A = \langle A_\mu, A_\nu \rangle$, then
(i) R(A) = $\langle R(A_\mu), R(A_\nu) \rangle$ and (ii) C(A) = $\langle C(A_\mu), C(A_\nu) \rangle$.

Consistency of xM=b, where M is Block Intuitionistic Fuzzy Matrix

In this section, we are concerned with a block intuitionistic fuzzy matrix of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (3.1)$$

with the diagonal blocks A and D are regular IFMs. With respect to this partitioning a schur complement of A in M is a matrix of the form M/A = D - CA^-B, Where A^- is some g-inverse of A. By M/A is a intuitionistic fuzzy matrix, we mean that CA^-B is invariant and D ≥ CA^-B. Therefore

$$M/A \text{ is a fuzzy matrix} \Leftrightarrow CA^-B \text{ is invariant and } D= D + CA^-B \quad (3.2)$$

Similarly,

$$M/D = A\text{-}BD^-C \text{ is an IFM} \Leftrightarrow BD^-C \text{ is invariant and } A=A + BD^-C \quad (3.3)$$

Let M of the form (3.1) can be expressed as M=<M_μ, M_ν>, where

$$M_\mu = \begin{pmatrix} A_\mu & B_\mu \\ C_\mu & D_\mu \end{pmatrix} \text{ and } M_\nu = \begin{pmatrix} A_\nu & B_\nu \\ C_\nu & D_\nu \end{pmatrix}$$

are block intuitionistic fuzzy matrices.

Let A = <A_μ, A_ν>, B = <B_μ, B_ν>, C = <C_μ,C_ν> and D =<D_μ, D_ν>.Since A and D are regular, by Lemma (2.2) A_μ, A_ν, D_μ, D_ν are all regular IFMs.

Lemma 3.1

For IFMs, A,B,C if A is regular. R(C) ⊆ R(A)and C(B) ⊆ C(A), then CA^-B is invariant for all choices of g-inverses of A.

Proof

Since R(C) ⊆ R(A) and C(B) ⊆ C(A), by lemma (2.1), C = YA and B = AX for some X,Y ∈ $(IF)_n$.

Now CA^-B = (YA) A^-(AX)
= Y (AA^-A)X (Since A is regular)
= YAX

Thus CA^-B is invariant for all choices of g-inverses of A.

Lemma 3.2

For IFMs A, B,C,if A is regular then the following are equivalent:

$R(C) \subseteq R(A)$

$R(C_\mu) \subseteq R(A_\mu)$, $R(C_\nu) \subseteq R(A_\nu)$

$C = CA^-A$ for all A^- of A.

$C_\mu = C_\mu A^-_\mu A_\mu$ for all A^-_μ of A_μ and

$C_\nu = C_\nu A^-_\nu A_\nu$ for all A^-_ν of A_ν.

Proof

Let $A = \langle A_\mu, A_\nu \rangle$, $B = \langle B_\mu, B_\nu)$ and $C = \langle C_\mu, C_\nu \rangle$.

(i) ⇔ (ii): Since $R(C) = \langle R(C_\mu), R(A_\nu) \rangle$ and $R(A) = \langle R(A_\mu), R(A_\nu) \rangle$.

$R(C) \subseteq R(A) \Leftrightarrow R(C_\mu) \subseteq R(A_\mu)$ and $R(C_\nu) \subseteq R(A_\nu)$. Thus (i) ⇔ (ii) holds.

(ii) ⇔ (iv): Since A is regular, by lemma (2.2), A_μ and A_ν are regular.

$R(C_\mu) \subseteq R(A_\mu) \Leftrightarrow C_\mu = Y_\mu A_\mu$ (By lemma (2.1))

$\Leftrightarrow C_\mu = Y_\mu (A_\mu A^-_\mu A_\mu)$

$\Leftrightarrow C_\mu = C_\mu A^-_\mu A_\mu$ (By taking $Y_\mu = C_\mu A^-_\mu$)

In the same manner $R(C_\nu) \subseteq R(A_\nu) \Leftrightarrow C_\nu = C_\nu A^-_\nu A_\nu$. Thus (ii) ⇔ (iv) holds.

(ii) ⇔ (iii):

Since A is regular, by lemma (2.2), A_μ, A_ν are regular.

$R(C_\mu) \subseteq R(A_\mu)$ and $R(C_\nu) \subseteq R(A_\nu)$

$\Leftrightarrow C_\mu = Y_\mu A_\mu$ and $C_\nu = Y_\nu A_\nu$ (By lemma (2.1))

$\Leftrightarrow C_\mu = Y_\mu A_\mu A^-_\mu A_\mu$ and $C_\nu = C_\nu A_\nu A^-_\nu A_\nu$

$\Leftrightarrow C_\mu = Y_\mu A^-_\mu A_\mu$ and $C_\nu = C_\nu A^-_\nu A_\nu$ (by taking $Y = CA^-$)

$\Leftrightarrow C = CA^-A$. Thus (ii) ⇔ (iii) holds.

Lemma 3.3

For IFMs A,B,C if A is regular then the following are equivalent.

$C(B) \subseteq C(A)$

$C(B_\mu) \subseteq C(A_\mu)$, $C(B_\nu) \subseteq C(A_\nu)$,

$B=AA^-B$ for all A^- of A.

$B_\mu = A_\mu A_\mu^- B_\mu$ for all A_μ^- of A_μ and

$B_\nu = A_\nu A_\nu^- B_\nu$ for all A_ν^- of A_ν

Proof

Since $C(B) = R(B^T)$, A is regular ⇔ A^T is regular and $A^- \in A\{1]$ ⇔$(A^-)^T \in A^T\{1\}$.

Lemma (3.3) follows from Lemma (3.2).

Theorem 3.4

For IFMs A, B, C if A is regular then the following are equivalent.

$R(C) \subseteq R(A)$ and $C(B) \subseteq C(A)$

$C = CA^-A$ and $B = AA^-B$ for all A^- of A

CA^-B is invariant, $C = C+CA^-A$ and $B = B+AA^-B$.

Proof

⇔ (iii): This is precisely the equivalence of (i) and (iii) of lemma (3.2) and lemma (3.3).

(ii) ⇒(iii): This follows from lemma(3.1) and (2.3), we get

$C = CA^-A$ and $B = AA^-B$

$\Rightarrow C = C + CA^-A$ and $B = B + AA^-B$

For any $A^- \in A\{1\}$, $C A^-B = (CXA) A^- (AYB)$, for some $X, Y \in A\{1\}$

$= C X (AA^-A) YB$

$= CXAYB$

$= C(XAY) B$

$= CZB$, Where $Z = XAY \in A\{1\}$.

Thus CA^-B is invariant

(iii) ⇒ (ii): $B = B+AA^-B$ and $C = C+ CA^-A$

$\Rightarrow B \geq AA^-B$ and $C \geq CA^-A$.

Suppose $C > CA^-A$, then $CA^-B > C(A^-AA^-) B = CXB$.

Where $X = A^-AA^-$ is a g-inverse of A. This contradicts the invariance of C.

Hence $C = CA^-A$.

In the same manner, we can show that $B = AA^-B$. Thus (ii) holds.

Here, we shall discuss the consistency of the intuitionistic fuzzy matrix equation $x M = b$.

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $x = [y\ z]$ and $b = [c\ d]$ be partitions of x and b respectively in confirmity with that of M.

Theorem 3.5

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with A and D are regular, M/A and M/D exists. $R(C) \subseteq R(A)$ and $R(B) \subseteq R(D)$. Then $x M = b$ is solvable if and only if $y.A = c$ and $z. D = d$ are solvable $c \geq dD^-C$ and $d \geq c A^-B$.

Proof

Suppose $x M = b$ is solvable and $\alpha = [\beta\gamma]$ is a solution. Then we get

$$\beta A + \gamma C = c \text{ and } \beta B + \gamma D = d \qquad (3.4)$$

Since $R(C) \subseteq R(A)$ and $R(B) \subseteq R(D)$, by theorem (3.4), $C = CA^-A$ and $B = BD^-D$.

Substituting for C and B in (3.4) we get

$\beta A + \gamma CA^-A = c$ and $\beta BD^-D + \gamma D = d$

$\Rightarrow (\beta+\gamma CA^-A)A = c$ and $(\beta BD^- + \gamma)D = d$.

Thus y.A = c, and z.D = d are solvable.

Since A and D are regular, the solutions will be of the form y= cA^- and z =dD^-

Hence $cA^- = \beta + \gamma CA^-$ and $dD^- = \beta BD^- + \gamma$

$\Rightarrow cA^-B = \beta B + \gamma CA^-B$ and $dD^-C = \beta BD^-C + \gamma C$ (3.5)

Since M/A and M/D exist, by (3.2) $A + BD^-C = A$ and $D + CA^-B = D$.

Substituting for A and D in (3.4) we get,

$c = \beta A + \gamma C = \beta(A + BD^-C) + \gamma C = \beta A + \beta BD^-C + \gamma C$ (3..6)

$d = \beta B + \gamma D = \beta B + \gamma(D + CA^-B) = \beta B + \gamma D + \gamma CA^-B$ (3.7)

Substituting (3.5) in (3.6) and (3.7), We get,

$c = \beta A + dD^-C$

$d = \gamma D + CA^-B$.

By intuitionistic fuzzy addition it follows that $C \geq dD^-C$ and $d \geq CA^-B$ as required.

Conversely,

Suppose y.A = c and z.C =d are solvable, then y = cA^- and z=dC^-. Hence cA^-A = y.A = c and $d\,C^-C$ = z.C = d.

From the given conditions, $c \geq dD^-C$ and $d \geq cA^-B$, We get

$c + dD^-C = c$ and $d + cA^-B = d$

Now, $[cA^-\ dD^-]\begin{pmatrix} A & B \\ C & D \end{pmatrix} = [cA^-A + dD^-C,\ cA^-B + dD^-D]$

$= [c + dD^-C,\ cA^-B + d]$

$= [c, d]$

Thus *x*.M=b is solvable. Hence the proof.

Remark 3.6

In particular, for B = 0 the above theorem reduces to the following

Corollary 3.7

For the matrix M = $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that $R(C) \subseteq R(A)$ the blocks A and D are regular IFMs, the following statements are equivalent:

x.M = b is solvable

y.A = c, z.D = d are solvable and $c \geq dD^-C$.

Proof

(i) ⇒ (ii): Suppose *x*.M = b is solvable, let $\alpha = [\beta\gamma]$ be a solution. Then we get,

$\beta.A + \gamma C = c$ and $\gamma D = d$. (3.8)

Since $R(C) \subseteq R(A)$, by theorem (3.4), $C = CA^{-}A$.Substituting C in (3.8) we get,
$\beta A + \gamma CA^{-}A = c$ and $\gamma D = d$
$\Rightarrow (\beta+\gamma CA^{-})A = c$ and $\gamma D = d$

Therefore y.A = c and z.D = d are both solvable with $y = \beta + \gamma CA^{-}$ is a solution of y.A = c and $z=\gamma$ is a solution of z.D=d. Since D is regular, $\gamma = dD^{-}$ is a solution of z.D = d. Now $\gamma C = dD^{-}C$. From $\beta A + \gamma C = c$ by addition property we get,$c \geq \gamma C = dD^{-}C$.

Hence (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i): Suppose y.A = C and z.D = d are solvable, since both A and D are regular IFMs y= cA^{-} and z = dD^{-} are respectively the solutions of the equation y.A=c and z.D=d. Hence $cA^{-}A=c$ and $dD^{-}D = d$. By addition property, $c \geq dD^{-}C$ implies $c + dD^{-}C = c$

$$[cA^{-}\ dD^{-}] \begin{pmatrix} A & O \\ C & D \end{pmatrix} = [cA^{-}A + dD^{-}C \quad dD^{-}D]$$
$$= [c+dD^{-}C \ \ d]$$
$$= [c \ \ d]$$
$$= b.$$

Thus $[cA^{-}\ dD^{-}]$ is a solution of the equation x.M = b. Hence x.M = b is solvable.

References

[1] H.H.Cho, Regular fuzzy matrices and fuzzy equations, Fuzzy Sets and Systems, 105(1999), 445 – 451.

[2] K.H.Kim and F.W.Roush, Generalised fuzzy matrices, Fuzzy sets and Systems. 4 (1980), 293 – 315.

[3] AR.Meenakshi, On Regularity of block fuzzy matrices, Int.J.Fuzzy Math., 12(2),(2004), 439-450.

[4] AR.Meenakshi, Fuzzy matrices: Theory and its applications, MJP Publishers, Chennai – 2008.

[5] AR.Meenakshi and T.Gandhimathi, On Regular intuitionistic Fuzzy matrices, Int.J. of Fuzzy Math., Vol. 19, No. 2, 2011.

[6] M.Pal, S.Khan and A.K.Shyamal, Intuitionistic fuzzy matrices, Notes on intuitionistic fuzzy sets 8(2), 2002, 51-62.