Global Journal of Mathematical Sciences: Theory and Practical.
ISSN 0974-3200 Volume 4, Number 2 (2012), pp.97–105

http://www.irphouse.com

Fixed point Theorems in partial Cone Metric Spaces

R. Krishnakumar

Department of Mathematics,
Urumu Dhanalakshmi College,
Tiruchirappalli-620019, Tamil Nadu,India.
E-mail: srksacet@yahoo.co.in

M. Marudai

Department of Mathematics,
Bharathidasan University,
Tiruchirappalli-620024, India.

Abstract

Let P be a subset of a Banach space E and P is normal and regular cone on E, we prove several fixed point theorems on partial cone metric spaces and these theorems generalize the recent results of varies authors.

AMS subject classification: Primary 47H10; Secondary 54H25.
Keywords: Partial Cone metric space, self mappings, Normal cone, regular cone, Fixed point.

1. Introduction and Preliminaries

In recent years, several authors (see [1,3,4,5,6]) have studied the strong convergence to a fixed point with contractive constant in cone metric spaces. In this paper we proved certain fixed point theorems in Partial cone metric spaces. We first recall definitions and known results that are needed in the sequel. Let E be a Banach space and a subset P of E is said to be a cone if it satisfies the following conditions,

(i) $P \neq \emptyset$ and P is closed;

(ii) $ax + by \in P$ for all $x, y \in P$ and a,b are non-negative real numbers;

(iii) $P \cap (-P) = \{0\}$.

The partial ordering $\leq$ with respect to the cone P by $x \leq y$ if and only if $y - x \in$ P. If $y - x \in$ interior of P, then it is denoted by $x \ll y$.The cone P is said to be a Normal if a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$. The cone P is called regular if every increasing sequence which is bounded above is convergent and every decreasing sequence which is bounded below is convergent.

Definition 1.1. Let X be a nonempty set, and the mapping $d : X \times X \to E$ is said to be a Cone metric space if it satisfies

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iii) $d(x, y) = d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Example 1.2. Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = R$ and $d : X \times X \longrightarrow E$ defined by

$$d(x, y) = (| x - y |, \alpha | x - y |)$$

where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space [1].

Definition 1.3. A partial cone metric on a nonempty set X is a function $p : X \times X \to E$ such that for all $x, y, z \in X$

(i) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$.

(ii) $0 \leq p(x, x) \leq p(x, y)$.

(iii) $p(x, y) = p(y, x)$.

(iv) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial cone metric space is a pair (X, p) such that X is a nonempty set and p is a partial cone metric on X. It is clear that, if $p(x, y) = 0$, then from (i) and (ii) $x = y$. But if $x = y$, $p(x, y)$ may not be 0.

A cone metric space is a partial cone metric space. But there are partial cone metric space which are not cone metric spaces. The following two examples illustrate such two partial cone metric spaces.

Example 1.4. Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = R^+$ and $p : X \times X \longrightarrow E$ defined by

$$p(x, y) = (\max\{x, y\}, \alpha \max\{x, y\})$$

where $\alpha \geq 0$ is a constant. Then (X, p) is a partial cone metric space which is not a cone metric space.

Example 1.5. Let $E = \ell_1$,

$$P = \{\{x_n\} \in \ell_1 : x_n \geq 0\},$$

Let $X = \{\{x_n\} : \{x_n\} \in R_n, \sum x_n < \infty\}$ where $R_n = \{$ set of all infinite sequences over $R^+\}$, and $p : X \times X \longrightarrow E$ defined by

$$p(x, y) = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n, \ldots\}$$

where the symbol $\vee$ denotes the maximum, i.e., $x \vee y = \max\{x, y\}$. Then (X, p) is a partial cone metric space which is not a cone metric space.

Let (X, p) be partial cone metric space, $x \in X$, and A be a non empty subset of X. we modify the concepts of distance between the set A and the singleton $\{x\}$, and the distance between two subsets A and B of X in the following:

$$p(x, A) = \inf\{p(x, a) : a \in A\}, \quad p(A, B) = \inf\{p(a, b) : a \in A, b \in B\}$$

Definition 1.6. Let (X, p) be a partial cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in intP$ there is N such that for all $n > N$. $p(x_n, x) \ll c + p(x, x)$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x \quad (n \to \infty)$$

Theorem 1.7. Let (X, p) be a partial cone metric space. P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $p(x_n, x) \to p(x, x) \quad (n \to \infty)$.

Proof. Suppose that $\{x_n\}$ converges to x. For every real $\epsilon > 0$, choose $c \in intP$ with $K\|c\| < \epsilon$. Then there is N, for all $n > N$, $p(x_n, x) \ll c + p(x, x)$. So that when $n > N$. $\|p(x_n, x) - p(x, x)\| \le K\|c\| < \epsilon$. This means $p(x_n, x) \to p(x, x) \quad (n \to \infty)$.

Conversely, suppose that $p(x_n, x) \to p(x, x) \quad (n \to \infty)$. For $c \in intP$, there id $\delta > 0$, such that $\|x\| < \delta$ implies $c - x \in intP$. For this δ there is N, such that for all $n > N$, $\|p(x_n, x) - p(x, x)\| < \delta$. So $c - |p(x_n, x) - p(x, x)| \in intP$. This means $p(x_n, x) - p(x, x) \ll c$. Therefore $\{x_n\}$ converges to x. ■

We note that let (X, p) be a partial cone metric space, P be a normal cone with normal constant K, if $p(x_n, x) \to p(x, x) \quad (n \to \infty)$ then $p(x_n, x_n) \to p(x, x) \quad (n \to \infty)$.

Lemma 1.8. Let $\{x_n\}$ be a sequence in partial cone metric space (X, p). If a point x is the limit of $\{x_n\}$ and $p(y, y) = p(y, x)$ then y is the limit point of $\{x_n\}$.

Proof. Suppose that $x_n \to x$ and $p(y, y) = p(y, x)$. Since for all $c \in intP$ there is N such that for all $n > N$

$$p(x_n, y) \le p(x_n, x) + p(x, y) - p(x, x) \ll c + p(y, y).$$

We have that $x_N \to y$. ■

Definition 1.9. Let (X, d) be a partial cone metric space $\{x_n\}$ be a sequence in X. $\{x_n\}$ is Cauchy sequence if there is $a \in P$ such that for every $\epsilon > 0$ there is N such that for all $n, m > N$

$$\|p(x_n, x_m) - a\| < \epsilon.$$

Theorem 1.10. Let (X, p) be a partial cone metric space. If $\{x_n\}$ is a Cauchy sequence in (X, p), then it is a Cauchy sequence in the cone metric space (X, d).

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X, p). There is $a \in P$ for every real $\epsilon > 0$ there is N, for all $n, m > N$ $\|p(x_n, x_m) - a\| < \frac{\epsilon}{4}$. Hence

$$d(x_n, x_m) = 2(p(x_n, x_m) - a) - (p(x_n, x_n) - a) - (p(x_m, x_m) - a)$$

So that when $n, m > N$ $\|d(x_n, x_m)\| < \epsilon$. This mean $d(x_n, x_m) \to 0 \quad (n, m \to \infty)$. ■

Definition 1.11. Let (X, p) is said to be a complete partial cone metric space if every Cauchy sequence is convergent.

In this paper we proved the theorems which are the generalization of the theorems of Huang Gaung,Zhang Xian [1].

2. Main results

Theorem 2.1. Let (X, d) be a complete partial cone metric space, P be a normal cone with normal constant K. Suppose that the mapping $T : \mathbf{X} \to \mathbf{X}$ satisfy the contractive condition

$$p(Tx, Ty) \leq ap(x, Tx) + bp(y, Ty)$$

for all $x, y \in X$, and $a + b < 1$, $a, b \in [0, 1)$. Then T has a unique fixed point in X. For each $x \in X$,the iterative sequence $\{T^n x\}_{n \geq 1}$ converges to the fixed point.

Proof. For every $x_0 \in X$ and $n \geq 1$, $Tx_0 = x_1$ and $Tx_n = x_{n+1} = T^{n+1} x_0$

$$\begin{aligned}
p(x_{n+1}, x_n) &= p(Tx_n, Tx_{n-1}) \\
&\leq ap(x_n, Tx_n) + bp(x_{n-1}, Tx_{n-1}) \\
&\leq ap(x_n, x_{n+1}) + bp(x_{n-1}, x_n) \\
p(x_{n+1}, x_n) &\leq Lp(x_n, x_{n-1}) \qquad \text{where} \quad L = \frac{b}{(1-a)} < 1 \\
p(x_{n+1}, x_n) &\leq L^n p(x_1, x_0)
\end{aligned}$$

For $n > m$ we have

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n-1}) + p(x_{n-1}, x_{n-2}) + \cdots \\ &\quad + p(x_{m+1}, x_m) - \sum_{k=1}^{n-m-1} p(x_{n-k}, x_{n-k}) \\ &\leq [L^{n-1} + L^{n-2} + ... + L^m] p(x_1, x_0) \\ &\leq L^m \frac{1 - L^{n-m}}{(1 - L)} p(x_1, x_0) \\ &\leq \frac{L^m}{(1 - L)} p(x_1, x_0) \end{aligned}$$

We get $\|p(x_n, x_m)\| \leq L^m K \dfrac{1}{1-L} \|p(x_1, x_0)\|$. Thus $\{T^n x\}$ is a Cauchy sequence in (X, p) such that $\lim_{n\to\infty} p(T^n x_0, T^m x_0) = 0$. As (X, p) is complete there exists $x_0 \in X$ such that $\{T^n x_0\}$ converges to x and

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n) = 0$$

Now for any $n \in N$, we have that,

$$\begin{aligned} p(Tx, x) &\leq p(Tx, T^{n+1}x_0) + p(T^{n+1}x_0, x) - p(T^{n+1}x_0, T^{n+1}x_0) \\ &\leq ap(x, Tx) + bp(T^n x_0, T^{n+1}x_0) + p(T^{n+1}x_0, x) \\ &\leq \frac{1}{1-a}[bp(T^n x_0, T^{n+1}x_0) + p(T^{n+1}x_0, x)] \\ \|p(Tx, x)\| &\leq \frac{K}{1-a}[b\|p(T^n x_0, T^{n+1}x_0)\| + \|p(T^{n+1}x_0, x)\|] \to 0. \end{aligned}$$

Hence $p(Tx, x) = 0$. But since

$$p(Tx, Tx) \leq ap(x, Tx) + bp(x, Tx) = (a + b)p(x, Tx) = 0$$

we have that $p(Tx, Tx) = p(Tx, x) = p(x, x) = 0$ which implies that $Tx = x$. So x is fixed point of T.

Now if y is another fixed point of T, then

$$p(x, y) = p(Tx, Ty) \leq ap(x, Tx) + bp(y, Ty) = 0$$

Hence $p(x, y) = p(x, x) = p(y, y) = 0$. We get $x = y$, thus the fixed point of T is unique. ∎

Corollary 2.2. Let (X, d) be a complete partial cone metric space, P be a normal cone with normal constant K. Suppose that the mapping $T : \mathbf{X} \to \mathbf{X}$ satisfy the contractive condition

$$p(Tx, Ty) \leq k[p(x, Tx) + p(y, Ty)]$$

for all $x, y \in X$, and $k \in (0, 1/2)$. Then T has a unique fixed point in X. For each $x \in X$,the iterative sequence $\{T^n x\}_{n\geq 1}$ converges to the fixed point.

Proof. The proof of the corollary immediate by taking $a = b = k$ in the above theorem. ■

We note that if we take p as a cone metric in the above theorem, then we obtain Theorem-3 of Huang Gaung,Zhang Xian [1] as a special case.

Theorem 2.3. Let (X, d) be a complete partial cone metric space, P be a normal cone on X. Suppose a mapping $T : \mathbf{X} \to \mathbf{X}$ satisfy the contractive condition

$$p(Tx, Ty) \leq r \max\{p(x, y), p(x, Tx), p(y, Ty)\}$$

for all $x, y \in X$ and $r \in [0, 1)$. Then T has a unique fixed point in X.

Proof. For every $x_0 \in X$ and $n \geq 1$, $Tx_0 = x_1$ and $Tx_n = x_{n+1} = T^{n+1}x_0$

$$\begin{aligned} p(x_{n+1}, x_n) &= p(Tx_n, Tx_{n-1}) \\ &\leq r \max[p(x_n, x_{n-1}), p(x_n, Tx_n), p(x_{n-1}, Tx_{n-1})] \\ &\leq r \max[p(x_n, x_{n-1}), p(x_n, x_{n+1}), p(x_{n-1}, x_n)] \\ &\leq rp(x_{n-1}, x_n) \\ &\leq r^n p(x_1, x_0) \end{aligned}$$

For $n > m$ we have

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n-1}) + p(x_{n-1}, x_{n-2}) + \cdots \\ &\quad + p(x_{m+1}, x_m) - \sum_{k=1}^{n-m-1} p(x_{n-k}, x_{n-k}) \\ &\leq [r^{n-1} + r^{n-2} + \cdots + r^m]p(x_1, x_0) \\ &\leq \frac{r^m}{(1-r)} p(x_1, x_0) \end{aligned}$$

We get $\|p(x_n, x_m)\| \leq K\dfrac{r^m}{(1-r)}\|p(x_1, x_0)\|$. Thus $\{T^n x\}$ is a Cauchy sequence in (X, p) such that $\lim\limits_{n\to\infty} p(T^n x_0, T^m x_0) = 0$. As (X, p) is complete there exists $x_0 \in X$ such that $\{T^n x_0\}$ converges to x and

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n) = 0$$

Now for any $n \in N$, we have that,

$$\begin{aligned} p(Tx, x) &\leq p(Tx, T^{n+1}x_0) + p(T^{n+1}x_0, x) - p(T^{n+1}x_0, T^{n+1}x_0) \\ &\leq r \max[p(x, T^n x_0), p(x, Tx), p(T^n x_0, T^{n+1}x_0)] + p(T^{n+1}x_0, x) \end{aligned}$$

$$\|p(Tx, x)\| \to 0$$

Hence $p(Tx, x) = 0$. But since

$$p(Tx, Tx) \leq r \max[p(x, x), p(x, Tx), p(x, Tx)] = 0$$

we have that $p(Tx, Tx) = p(Tx, x) = p(x, x) = 0$ which implies that $Tx = x$. So x is fixed point of T.

Now if y is another fixed point of T, then

$$p(x, y) = p(Tx, Ty) r \max[p(x, y), p(x, Tx), p(y, Ty)] = 0$$

Hence $p(x, y) = p(x, x) = p(y, y) = 0$. We get $x = y$, thus the fixed point of T is unique. ■

Corollary 2.4. Let (X, p) be a complete partial cone metric space, P be a normal cone with normal constant K. Suppose that the a mapping $T : \mathbf{X} \to \mathbf{X}$ satisfy the contractive condition

$$p(Tx, Ty) \leq cp(x, y),$$

for all $x, y \in X$, where $c \in (0, 1)$ is a constant. Then T has a unique fixed point in X, and for any $x \in X$ iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof. The proof of the corollary immediately follows by taking $p(x, y)$ as maximum value and $r = c$ in the previous theorem. ■

Theorem 2.5. Let (X, d) be a complete partial cone metric space, P be a normal cone on X. Suppose a mapping $T, S : \mathbf{X} \to \mathbf{X}$ satisfy the contractive condition

$$p(Tx, Sy) \leq a[p(x, Tx) + p(y, Sy)]$$

for all $x, y \in X$ and $a \in (0, 1/2)$. Then T and S have a unique common fixed point in X.

Proof. For every $x_0 \in X$ and $n \geq 1$, $Tx_0 = x_1$ and $Sx_1 = x_2$

$$\begin{aligned} p(x_1, x_2) &= p(Tx_0, Sx_1) \\ &\leq a[p(x_0, Tx_0) + p(x_1, Sx_1)] \\ &\leq a[p(x_0, x_1) + p(x_1, x_2)] \\ &\leq \left(\frac{a}{1-a}\right) p(x_0, x_1) \end{aligned}$$

which implies that

$$p(x_1, x_2) \leq kp(x_0, x_1),$$

where $0 < k = \left(\dfrac{a}{1-a}\right) < 1$. Now for $x_2 = Sx_1$ there exists $x_3 = Tx_2$ such that $p(x_2, x_3) \leq kp(x_1, x_2)$. Continuing this process we obtain a sequence $\{x_n\}$ in X with

$x_{2n-1} = Tx_{2n-2}$, $x_{2n} = Sx_{2n-1}$ such that $p(x_n, x_{n+1}) \leq kp(x_{n-1}, x_n)$ which further implies that $p(x_n, x_{n+1}) \leq k^n p(x_0, x_1)$ for all $n \geq 1$. Then, for $n > m$ we have

$$\begin{aligned}
p(x_n, x_m) &\leq p(x_n, x_{n-1}) + p(x_{n-1}, x_{n-2}) + \cdots \\
&\quad + p(x_{m+1}, x_m) - \sum_{k=1}^{n-m-1} p(x_{n-k}, x_{n-k}) \\
&\leq [k^{n-1} + k^{n-2} + \cdots + k^m] p(x_1, x_0) \\
&\leq \left(\frac{k^m}{1-k}\right) p(x_1, x_0)
\end{aligned}$$

We get $\|p(x_n, x_m)\| \leq K \left(\frac{k^m}{1-k}\right) \|p(x_1, x_0)\|$. This implies $p(x_n, x_m) \to 0$ $n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence in (X, p). By the completeness of X, there $x \in X$ such that $x_n \to x$, $n \to \infty$.

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n) = 0$$

Now for any $n \in N$, we have that,

$$\begin{aligned}
p(Tx, x) &\leq p(Tx, x_{2n}) + p(x_{2n}, x) - p(x_{2n}, x_{2n}) \\
&\leq p(Tx, Sx_{2n-1}) + p(x_{2n}, x) \\
&\leq a[p(x, Tx) + p(x_{2n-1}, x_{2n})] + p(x_{2n}, x) \\
&\leq k[ap(x_{2n-1}, x_{2n}) + p(x_{2n}, x)] \\
\|p(Tx, x)\| &\to 0
\end{aligned}$$

Hence $p(Tx, x) = 0$. Similarly, $p(Sx, x) = 0$. $p(Tx, Sx) \leq a[p(x, Tx) + p(x, Sx)] = 0$, and also $p(Tx, Tx) = p(Sx, Sx) = 0$. This implies that $p(Tx, Tx) = p(Tx, x) = p(x, x) = 0$ and $p(Sx, Sx) = p(Sx, x) = p(x, x) = 0$. Hence $Tx = Sx = x$. So x is the common fixed point of S, T. ■

References

[1] Huang Gaung, Zhang Xian, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, 332:1468–1476, 2007.

[2] S.G. Matthews, Partial metric topology, *Annals of The New year Academy of Science*, 183–197, 1994.

[3] H. Mohebi, Topical functions and their properties in a class of ordered Banach spaces, in continuous Optimization, *Current Trends and Modern Applications*, Part II, Springer, pp. 343–361, 2005.

[4] H. Mohebi, H. Sadeghi, A.M. Rubinov, Best approximation in a class of normed spaces with star-shaped cone, *Numer. Funct. Anal. Optim.*, 27(3-4):411–436, 2006.

[5] H. Lakzian, F. Arabyani, Some Fixed Point Theorems in Cone Metric Spaces with w-Distance, *Int. Journal of Math. Analysis*, 3(22):1081–1086, 2009.

[6] Sh. Rezapour, R. Hamlbarani, Some notes on the Cone metric Spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, 345:719–724, 2008.