Global Journal of Mathematical Sciences: Theory and Practical.
ISSN 0974-3200 Volume 4, Number 2 (2012), pp. 107–112

http://www.irphouse.com

Some Fixed Point Theorems for Expansive type Mappings in Dislocated Metric Space

Abha Tenguria
Govt. M. L. B. Girls P. G. College Bhopal (M.P.) India
E-mail: ten.abha@gmail.com

Prabha Chouhan
Scope College of Engineering Bhopal (M.P.) India
E-mail: prabhachouhan@yahoo.com

Abstract

The objective of this paper is to obtain some fixed point theorems for Dislocated metric space for generalized contraction mappings.

AMS subject classification: 47H10, 54H25.
Keywords: Dislocated metric, Fixed point.

1. Introduction

In recent years, the study of fixed points in dislocated metric space have attracted much attention, some of the recent literatures in dislocated metric space may be noted in [1, 2, 3, 4] in this paper we construct a sequence of points and consider its convergence to the fixed point of continuous mapping defined on dislocated metric space. For the purpose of obtaining the fixed point, we have used rational inequality.

2. Preliminaries

Definition 2.1. Let X be a nonempty set let $d : X \times X \to [0, \infty)$ be a function satisfying following condition.

(i) $d(x, y) = d(y, x) = 0$ implies $x = y$,

(ii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a dislocated quasi-metric on X. If d satisfying $d(x, x) = 0$, then it is called a quasi-metric on X. If d satisfies $d(x, y) = d(y, x)$,then it is called a dislocated metric.

Definition 2.2. A sequence $\{x_n\}$ in dislocated metric space (X, d) is called Cauchy if for given $\varepsilon > 0$, $\exists\, n_0 \in N$ such that $\forall\, m, n \geq n_0$, implies $d(x_m, x_n) < \varepsilon$.

Definition 2.3. A Sequence $\{x_n\}$ dislocated converges to x if

$$\lim_{n\to\infty} d(x_n, x) = 0$$

In this case x is called a limit of $\{x_n\}$ and write $x_n \to x$.

Definition 2.4. A dislocated metric space (X, d) is called complete if every Cauchy sequence converges in it.

3. Main Results

Theorem 3.1. Let (X, d) be a complete dislocated metric space. Let $T : X \to X$ be continuous mapping satisfies.

$$d(Tx, Ty) \geq \frac{a[d(x, Tx)d(x, y) + d(y, Ty)d(x, y)] + bd(x, Tx)d(y, Ty) + c[d(x, y)]^2}{d(x, Tx) + d(y, Ty) + d(x, y)} \tag{3.1}$$

for each $x, y \in X$, $x \neq y$ where $a, b, c \geq 0$, $2a + b + c > 3$ and $c > 1$. Then T has a unique fixed point.

Proof. Let x_o be an arbitrary point in X, there is x_1 in X such that $T(x_1) = x_o$. In this way we define a sequence $\{x_n\}$ as follows.

$$x_n = Tx_{n+1} \quad \text{for } n = 0, 1, 2, \tag{3.2}$$

If $x_n = x_{n+1}$ for some n then we see that x_n is a fixed point of T, therefore we suppose that no two consecutive terms of sequence $\{x_n\}$ are equal.

Now consider

$$d(x_n, x_{n+1}) = d(Tx_{n+1}, Tx_{n+2})$$

$$\geq \frac{\begin{array}{c} a[d(x_{n+1}, Tx_{n+1})d(x_{n+1}, x_{n+2}) + d(x_{n+2}, Tx_{n+2})d(x_{n+1}, x_{n+2})] \\ +b\ d(x_{n+1}, Tx_{n+1})d(x_{n+2}, Tx_{n+2}) + c\ d^2(x_{n+1}, x_{n+2}) \end{array}}{d(x_{n+1}, Tx_{n+1}) + d(x_{n+2}, Tx_{n+2}) + d(x_{n+1}, x_{n+2})}$$

$$= \frac{\begin{array}{c} a[d(x_{n+1}, x_n)d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+1})d(x_{n+1}, x_{n+2})] \\ +b\ d(x_{n+1}, x_n)d(x_{n+2}, x_{n+1}) + c\ d^2(x_{n+1}, x_{n+2}) \end{array}}{d(x_{n+1}, x_n) + d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_{n+2})}$$

$$\Rightarrow d(x_n, x_{n+1})[d(x_n, x_{n+1}) + 2d(x_{n+1}, x_{n+2})]$$
$$\geq (2a + b + c)d(x_{n+1}, x_{n+2}) \min\{d(x_{n+1}, x_n)d(x_{n+1}, x_{n+2})\}$$
$$\Rightarrow d^2(x_n, x_{n+1}) \geq (2a + b + c - 2)d(x_{n+1}, x_{n+2})$$
$$\min\{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})\}$$

Case I

$$d^2(x_n, x_{n+1}) \geq (2a + b + c - 2)\ d^2(x_{n+1}, x_{n+2})$$
$$\Rightarrow d(x_{n+1}, x_{n+2}) \leq \left(\frac{1}{2a + b + c - 2}\right)^{1/2} d(x_n, x_{n+1})$$
$$\Rightarrow d(x_{n+1}, x_{n+2}) \leq k_1\ d(x_n, x_{n+1})$$

where $k_1 = \left(\frac{1}{2a + b + c - 2}\right)^{1/2} < 1$ *[As* $2a + b + c > 3$*]*

Case II

$$d^2(x_n, x_{n+1}) \geq (2a + b + c - 2)d(x_{n+1}, x_{n+2})\ d(x_n, x_{n+1})$$
$$\Rightarrow d(x_{n+1}, x_{n+2}) \leq \frac{1}{(2a + b + c - 2)}\ d(x_n, x_{n+1})$$
$$\Rightarrow d(x_{2n+1}, x_{2n+2}) \leq k_2\ d(x_n, x_{n+1})$$

where $k_2 = \frac{1}{(2a + b + c - 2)} < 1$ *[As* $(2a + b + c) > 3$*]*
Let $k = max\{k_1, k_2\} < 1$.
So, in general

$$d(x_n, x_{n+1}) \leq k\ d(x_{n-1}, x_n) \text{ for } n = 1, 2, 3., \ldots$$
$$\Rightarrow d(x_n, x_{n+1}) \leq k^n\ d(x_0, x_1) \tag{3.3}$$

Now we shall prove that $\{x_n\}$ is a Cauchy sequence. For this for every positive integer p, we have

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\leq (k^n + k^{n+1} + k^{n+2} + \cdots + k^{n+p-1}) d(x_0, x_1) \ \textit{[by 3.1.3]} \\ &= k^n(1 + k + k^2 + \cdots + k^{p-1})\, d(x_0, x_1) \\ &< \frac{k^n}{(1-k)} d(x_0, x_1) \end{aligned}$$

As $n \to \infty$, $d(x_n, x_{n+p}) \to 0$, it follows $\{x_n\}$ is a Cauchy sequence in X. As X is a complete dislocated metric space, so there exist a point $u \in X$ such that $\{x_n\} \to u$. Since T is a continuous, so

$$T(u) = T(\lim x_n) = \lim T(x_n) = \lim x_{n+1} = u.$$

Thus u is a fixed point of T.

Uniqueness: Let z be another fixed point of T, that is $Tz = z$

$$\begin{aligned} d(x, z) &= d(Tx, Tz) \\ &\geq \frac{ad(x, Tx)\ d(x, z) + d(z, Tz)[ad(x, z) + bd(x, Tx)] + c[d(x, y)]^2}{d(x, Tx) + d(z, Tz) + d(x, z)} \\ &= \frac{ad(x, Tx)d(x, z) + d(z, z)[ad(x, z) + bd(x, x)] + c[d(x, z)]^2}{d(x, x) + d(z, z) + d(x, z)} \\ &= c\ d(x, z) \\ &\Rightarrow d(x, z) \leq \frac{1}{c} d(x, z) \ \textit{[as } c > 1\textit{]} \\ &\Rightarrow d(x, z) = 0 \\ &\Rightarrow x = z \end{aligned}$$

This completes the proof of the theorem 3.1. ■

Theorem 3.2. Let (X, d) be a complete dislocated metric space. Let $T : X \to X$ be continuous mapping satisfies.

$$d(Tx, Ty) \geq \frac{ad(x, Tx)[1 + d(y, Ty)]}{1 + d(x, y)} \tag{3.4}$$

$$+ b[d(x, Tx) + d(y, Ty)] + cd(x, y) \tag{3.5}$$

for each $x, y \in X$, $x \neq y$, where $a, b \geq 0$, and $c > 1$. Then T has a unique fixed point.

Proof. Construct a sequence $\{x_n\}$ as in proof of theorem 3.1. We claim that the inequality (3.1.4) for $x = x_{n+1}$ and $y = x_{n+2}$ implies that.

$$\begin{aligned}
d(Tx_{n+1}, Tx_{n+2}) &\geq \frac{ad(x_{n+1}, Tx_{n+1})[1 + d(x_{n+2}, Tx_{n+2})]}{1 + d(x_{n+1}, x_{n+2})} \\
&\quad + b\,[d(x_{n+1}, Tx_{n+1}) + d(x_{n+2}, Tx_{n+2})] + c\; d(x_{n+1}, x_{n+2}) \\
\Rightarrow d(x_n, x_{n+1}) &\geq \frac{ad(x_{n+1}, x_n)[1 + d(x_{n+2}, x_{n+1})]}{1 + d(x_{n+1}, x_{n+2})} \\
&\quad + b\,[d(x_{n+1}, x_n) + d(x_{n+2}, x_{n+1})] + c\; d(x_{n+1}, x_{n+2}) \\
\Rightarrow d(x_{n+1}, x_{n+2}) &\leq \frac{[1 - (a + b)]}{(b + c)} d(x_n, x_{n+1}) \\
\Rightarrow d(x_{n+1}, x_{n+2}) &\leq kd(x_n, x_{n+1})
\end{aligned}$$

where

$$\begin{aligned}
k = \left[\frac{1 - (a + b)}{(b + c)}\right] &< 1 \text{ [Since } c > 1 \\
\Rightarrow\; a + 2b + c\; &> 1 \Rightarrow k < 1]
\end{aligned}$$

In general

$$\begin{aligned}
&\Rightarrow d(x_n, x_{n+1}) \leq k\; d(x_{n-1}, x_n) \quad \text{for } n = 1, 2, 3, \ldots \\
&\Rightarrow d(x_n, x_{n+1}) \leq k^n\; d(x_0, x_1) \qquad (3.6)
\end{aligned}$$

We can prove that $\{x_n\}$ is a Cauchy sequence using (3.1.5) as proved in theorem 3.1 and since X is a complete dislocated metric space, so there exists a point u in X such that

$$\{x_n\} \to u \in X.$$

Since T is a continuous, so

$$T(u) = T(\lim x_n) = \lim T(x_n) = \lim x_{n+1} = u.$$

Thus u is a fixed point of T.

Uniqueness: Let z be another fixed point of T, that is $Tz = z$

$$\begin{aligned}
d(x, z) &= d(Tx, Tz) \\
&\geq \frac{ad(x, Tx)[1 + d(z, Tz)]}{1 + d(x, z)} + b[d(x, Tx) + d(z, Tz)] + c\; d(x, z) \\
&= \frac{ad(x, x)[1 + d(z, z)]}{1 + d(x, z)} + b[d(x, x) + d(z, z)] + c\; d(x, z) \\
\Rightarrow d(x, z) &\leq \frac{1}{c} d(x, z) \\
\Rightarrow d(x, z) &= 0\; [As c > 1] \\
\Rightarrow x &= z \qquad (3.7)
\end{aligned}$$

This completes the proof of the theorem 3.2. ■

References

[1] A. Isufati, Fixed point Theorems in Dislocated Quasi Metric Space. *Appl. Math. Sci.*, 4(5):217–223, 2010.

[2] C.T. Aage, and J.N. Salunke, The Results on Fixed point Theorems in Dislocated and Dislocated Quasi Metric Space. *Appl. Math. Sci.*, 2(59):2941–2948, 2008.

[3] F.M. Zeyada, G.H. Hassan, and M.A. Ahmed, A Generalization of Fixed point Theorem Due to Hitzler & Seda in Dislocated Quasi Metric Space, *The Arabian Journal for Science & Engeering*, 31:111–114, 2005.

[4] Haimabati Chatterjee, On Generalization of Banach Contraction Principal, *Indian J. Pure Appl. Math.*, 10(4):400–403, 1979.