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On Mappings Admitting Centers

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Abstract

In this paper we discuss in metric spaces a class of mappings admitting centers, which in some sense, include the (quasi) nonexpansive mappings. This class of mappings in Banach spaces was introduced by J. Garcia-Falset, E. Llorens-Fuster and S. Prus [Nonlinear Analysis 66 (2007), 1257–1274]. The results proved in this paper generalize and extended some of the results proved by J. Garica-Falset et al. and of T. Dominguez Benavides, J. Garcia-Falset, E. Llorens-Fuster and P. Lorenzo Ramirez [Nonlinear Analysis 71 (2009), 1562–1571].

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Suppose C is a subset of a metric space (X, d) and $T : C \to X$ a nonexpansive mapping with a fixed point $y_0 \in C$, then for every $x \in C$,

$$d(Tx, y_0) \leq d(x, y_0) \tag{1}$$

This inequality may be satisfied even for a nonexpansive fixed point free mapping. For example (see [4]), consider the affine Beal's mapping

$$T(x_1, x_2, \ldots) = (1, x_1, x_2, \ldots)$$

defined on the unit ball B of the classical sequence space c_0 of all convergent sequences converging to 0 with supremum norm. Take $y_0 = (2, 0, 0, \ldots)$.It can be easily seen that T satisfies inequality (1) in B , as for every $x = (x_1, x_2, \ldots) \in B$

$$\|T(x) - y_0\| = \|(-1, x_1, x_2, \ldots)\| = 1 \leq 2 - x_1 = \|(x_1 - 2, x_2, \ldots)\| = \|x - y_0\|.$$

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Although y_0 is not a fixed point of the nonexpansive mapping T, but $y_0 \in c_0$ is such that inequality (1) holds. We shall call such a point y_0 as a center for T. This notion of center of a mapping was introduced and discussed in Banach spaces in [4] and subsequently, this study was taken up in [3]. We intend to study the class of mappings admitting centers in spaces more general than normed linear spaces. This class contains contraction mappings, quasi nonexpansive mappings (i.e. those for which every fixed point is a center) which properly contains the class of nonexpansive mappings having fixed points, although there are non quasi nonexpansive mappings that admit a center (see [4]).

If C is a nonempty subset of a metric space (X, d) and $y_0 \in X$, a point $x_0 \in C$ is called a **nearest point** or a **best approximant** to y_0 in C if

$$d(y_0, x_0) = dist(y_0, C) \equiv inf\{d(y_0, x) : x \in C\}$$

. The set of all nearest points in C to y_0 is denoted by $P_C(y_0)$. The (set-valued) mapping P_C is called the **nearest point mapping**. C is said to be **proximinal** if $P_C(y_0) \neq \phi$ for each $y_0 \in X$, and is called **Chebyshev** if $P_C(y_0)$ is exactly a singleton for every $y_0 \in X$.

A subset C of a metric space (X, d) is said to be **approximatively compact** (see [12], p.382) if for every $x \in X$ and every sequence $< y_n >$ in C satisfying $limd(x, y_n) = dist(x, C)$ there exists a subsequence $< y_{n_i} >$ converging to an element of C.

Let (X, d) be a metric space. A continuous mapping $W : X \times X \times [0, 1] \to X$ is said to be a **convex structure** on X , if for $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

holds for all $u \in X$.

The triplet (X, d, W) is called a **convex metric space** [13]. A normed linear spaces and each of its convex subsets are simple examples of convex metric spaces. There are many convex metric spaces which are not normed linear spaces (see e.g. [6], [13]).

A subset C of a convex metric space (X, d, W) is said to be **convex** [13] if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$.

A convex metric space (X, d, W) is said to be **uniformly convex** [11] if for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leq r$, $d(z, y) \leq r$ and $d(x, y) \geq r\varepsilon$, $d(z, W(x, y, 1/2)) \leq r(1 - \delta) < r$.

It is easy to prove (see [1]) that if $< x_n >$ and $< y_n >$ are sequences in a uniformly convex metric space (X, d, W) such that for some $z \in X$, $\lim d(z, x_n) = \lim d(z, y_n) = r$ and $\lim d(z, W(x_n, y_n, \lambda)) = r$ then $\lim d(x_n, y_n) = 0$.

A metric space (X, d) is called a **linear metric space** if (i) X is a linear space, (ii) addition and scalar multiplication in X are continuous, and (iii) d is translation invariant i.e. $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in X$.

A mapping $T : C \to X$ is said to be (i) **nonexpansive** if, $d(Tx, Ty) \leq d(x, y)$ for each $x, y \in C$, (ii) **contraction** if there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for each $x, y \in C$, (iii) **quasi-nonexpansive** if T has atleast one fixed point $y_0 \in C$ and $d(Tx, y_0) \leq d(x, y_0)$ for each $x \in C$.

A mapping $T : C \to C$ is said to be **asymptotically regular** if $\lim d(T^n x, T^{n+1} x) = 0$ for all $x \in C$.

Let C be a nonempty subset of a metric space (X, d). A point $y_0 \in X$ is called a **center** for the mapping $T : C \to X$ if, for each $x \in C$

$$d(Tx, y_0) \le d(x, y_0).$$

Remarks 1. (see [4]) If $T : C \to X$ has a center $y_0 \in C$, then trivially $T(y_0) = y_0$. Thus fixed point results for mappings admitting centers are only nontrivial provided they have a center $y_0 \notin C$.

It may be pointed out that $T : C \to C$ is quasi-nonexpansive provided that T has at least one fixed point in C and every fixed point is a center for T.

A point $y_0 \in X$ is called a **strict center** for the mapping $T : C \to X$ if, for each $x \in C$ such that $x \ne T(x)$, $d(Tx, y_0) < d(x, y_0)$.

We start with some consequences of the definition of center for a mapping

Proposition 2. If C is a convex subset of convex metric space (X, d, W), $T : C \to X$ has a center $y_0 \in X$ and $r \in (0, 1)$, then the mapping T_r defined by $T_r(x) = W(x, T(x), r)$ has the same center y_0.

Proof. Consider $d(T_r(x), y_0) = d(W(x, T(x), r), y_0)$

$$\begin{aligned} &\le rd(x, y_0) + (1 - r)d(Tx, y_0) \\ &\le rd(x, y_0) + (1 - r)d(x, y_0) \\ &= d(x, y_0) \end{aligned}$$

for each $x \in C$. Hence y_0 is a center of T_r. ■

Remarks 3. The above result is analogous to the result (see [1]):

Let $T : C \to C$ be a mapping on a convex subset C of a convex metric space X then the mapping T_r has the same fixed points as T.

We also have: If x is a fixed point of T_r then x is also a fixed point of T. It is not known whether an analog of this is true in the case of center for a mapping.

Corollary 4. 4If C is a convex subset of a normed linear space X and a mapping $T : C \to X$ has a center $y_0 \in X$ then the mapping $T_r = rI + (1 - r)T$ has the same center y_0.

Proposition 5. If C is a subset of a metric space (X, d) and $T : C \to C$ has a center $y_0 \in X$, then all its iterates $T^n : C \to C$ have the same center y_0.

Proof. Consider $d(T^n x, y_0) = d(T(T^{n-1}x), y_0)$

$$\begin{aligned} &\le d(T^{n-1}x, y_0) \\ &\le d(T^{n-2}x, y_0) \\ &\le d(x, y_0). \end{aligned}$$

■

Proposition 6. If a mapping $T : C \to X$ has a center y_0, then so does the restriction of T to every subset of C, where C is a subset of a metric space (X, d).

Proof. The proof is obvious. ■

Proposition 7. If C is a subset of a linear metric space (X, d) and $T : C \to X$ has a center $y_0 \in X$, then the mapping $\hat{T} : C - \{y_0\} \to X$ given by $\hat{T}(x - y_0) = T(x) - y_0$ admits the center $0 \in X$.

Proof. Consider
$$\begin{aligned} d(\hat{T}(x - y_0), 0) &= d(T(x) - y_0, 0) \\ &= d(Tx, y_0) \\ &\le d(x, y_0) \\ &= d(x - y_0, 0) \end{aligned}$$

for each $x \in C$ and so the result follows. ■

Notes 1: Propositions 5, 6 and 7 are known in normed linear spaces (see [4]).
2(i). Although each fixed point of a nonexpansive mapping is its center, for Lipschitzian mappings this need not be true (see [4]).
(ii) There are expansive and non-Lipschitzian mappings admitting centers (see [4]). For such mappings, a fixed point may or may not be their centers.
(iii) A mapping may have one or more centers or even no center (see [4]).

As already pointed out, every nonexpansive mapping $T : C \to X$ having a fixed point is a continuous mapping admitting centers (every fixed point of such a mapping is a center). It is well known (see [14]) that the fixed point set of a nonexpansive mapping is always closed. The following proposition shows that same is true for the set of centers.

Proposition 8. If C is a subset of a metric space (X, d) and $T : C \to X$ is a mapping then the set $Z(T) = \{y_0 \in X : d(Tx, y_0) \le d(x, y_0)$ for each $x \in C\}$ of all centers of T is a closed set.

Proof. If $Z(T) = \phi$, the result is obvious. So, suppose $Z(T) \neq \phi$. Let $z \in \overline{Z(T)}$. Then there exists a sequence $z_n \in Z(T)$ such that $z_n \to z$. Consider

$$\begin{aligned} d(Tx, z) = d(Tx, lim z_n) = \lim d(Tx, z_n) &\le \lim d(x, z_n) \\ &= d(x, lim z_n) \\ &= d(x, z) \end{aligned}$$

for each $x \in C$. Consequently, $z \in Z(T)$ and so $Z(T)$ is closed. ■

Remarks 9. It is well known (see [5]) that the fixed point set of a nonexpansive (quasi-nonexpansive) mapping need not be convex and it is convex only if we impose certain conditions on the set C and/or on the space X (see e.g. [5], [11]). It will be interesting to discuss the convexity of $Z(T)$ in various abstract spaces.

The following proposition gives a sufficient condition for the convergence of a sequence of iterates for mappings admitting centers.

Proposition 10. Let C be a closed subset of a complete metric space (X, d) and $T : C \to X$ be a continuous mapping admitting center. Suppose there exists $x_0 \in C$ such that for every $n \geq 0$, $T^n(x_0) \in C$. If $\lim dist(T^n x_0, Z(T)) = 0$ then T has a fixed point x^* and the sequence of Picard's iterates$\{T^n(x_0) : n \in \mathbb{N}\}$ converges to x^*.

Proof. Firstly, we show that $< T^n x_0 >$ is a Cauchy sequence. Let $\varepsilon > 0$ be given.Consider $\varepsilon/2 > 0$, there exists a positive integer n_0 such that for all $n \geq n_0$, $dist(T^n x_0, Z(T)) < \varepsilon/2$. This implies that there exists $z_0 \in Z(T)$ such that $d(T^{n_0} x_0, Z_0) < \varepsilon/2$. Since z_0 is a center for T, for $m, n \geq n_0$ we have

$$\begin{aligned} d(T^n x_0, T^m x_0) &\leq d(T^n x_0, z_0) + d(z_0, T^m x_0) \\ &\leq d(T^{n_0} x_0, z_0) + d(T^{n_0} x_0, z_0) \\ &< \varepsilon. \end{aligned}$$

Therefore $< T^n x_0 >$ is a Cauchy sequence in C. Since C being closed, is complete, $< T^n x_0 > \to x^* \in C$.

Now we show that x^* is a fixed point of T. Since $\lim dist(T^n x_0,\ Z(t)) = 0$, $dist(x^*, Z(t)) = 0$. This gives $x^* \in \overline{Z(T)} = Z(T)$ i.e. x^* is a center for T which also lies in C and hence x^* is a fixed point of T. ■

Remarks 11. In the light of Proposition 2, we see that Proposition 10 holds if we replace T by $T_r (0 < r < 1)$ and, we assume that C is a convex subset of a convex metric space (X, d, W).

For Banach spaces, Proposition 12 was proved by G. Garcia-Falset et al. (see [4]-Proposition 10).

In the case of Banach spaces, it was shown (see [7]) that if $T : C \to C$ is nonexpansive then the mappings T_r are asymptotically regular. We now obtain a similar property for continuous mappings admitting centers in the case of uniformly convex metric spaces.

Proposition 12. Let C be a closed convex subset of a uniformly convex metric space (X, d, W). If $T : C \to C$ is a mapping admitting a center, then every mapping $T : C \to C$ is asymptotically regular.

Proof. Let $z \in X$ be a center for T. Then by Proposition 1, z is also a center for T_r, $0 < r < 1$. We claim that for each $x \in C$, the sequence $< \beta_n >$ defined by $< \beta_n >= d(T_r^n x, z)$ is a decreasing sequence. Let $x_0 \in C$ be arbitrary. Consider

$$d(T_r^{n+1} x_0, z) = d(T(T_r^n x_0), z) \leq d(T_r^n x_0, z).$$

Since $< \beta_n >$ is a decreasing sequence of positive real numbers, it converges to some β, i.e.

$$\lim_{n\to\infty} d(T_r^n x_0, z) = \beta \tag{1}$$

If $\beta = 0$, the conclusion is obvious as

$$\lim_{n\to\infty} d(T_r^{n+1}x_0, T_r^n x_0) \leq \lim_{n\to\infty} d(T_r^{n+1}x_0, z) + \lim_{n\to\infty} d(z, T_r^n x_0).$$

So, assume that $\beta > 0$. Now (1) also implies

$$\lim_{n\to\infty} d(T_r^{n+1}x_0, z) = \beta \tag{2}$$

i.e.

$$\lim_{n\to\infty} d(T_r(T_r^n x_0), z) = \beta$$

i.e.

$$\lim_{n\to\infty} d[W(T_r^n x_0, T(T_r^n x_0, r), z] = \beta \tag{3}$$

Since (X, d, W) is uniformly convex, (1), (2) and (3) imply

$$\lim_{n\to\infty} d(T_r^n x_0, T(T_r^n x_0)) = 0 \tag{4}$$

Consider

$$\lim_{n\to\infty} d(T_r^n x_0, T_r^{n+1}x_0) = \lim_{n\to\infty} d[T_r^n x_0, W(T_r^n x_0, T(T_r^n x_0), r)]$$

$$\leq (1-r) \lim_{n\to\infty} d(T_r^n x_0, T(T_r^n x_0)) = 0.$$

Hence T_r is asymptotically regular on C. ■

Remarks 13. For uniformly convex Banach spaces, this result was proved by J. Garcia-Falset et al. ([4]-Proposition 12).

Theorem 14. Let (X, d, W) be a convex metric space and C a nonempty closed convex subset of X such that $P_C(x)$ is nonempty and compact for each $x \in X$. Then every continuous mapping $T : C \to C$ admitting a center in X has a fixed point.

Proof. Let $x_0 \in X$ be a center of the continuous mapping $T : C \to C$. If $x_0 \in C$ then x_0 is a fixed point of T. Suppose $x_0 \notin C$. Consider the set $P_C(x_0)$. Since C is a convex subset of the convex space X, $P_C(x_0)$ is convex (see [10]). Therefore $P_C(x_0)$ is a non-empty compact convex subset of the convex space X. We claim that $P_C(x_0)$ is T-invariant. Let $y \in P_C(x_0)$. Consider

$$d(x_0, C) \leq d(x_0, Ty) \leq d(x_0, y) = d(x_0, C).$$

This gives $Ty \in P_C(x_0)$.Thus T is a continuous self mapping of a nonempty compact convex subset $P_C(x_0)$ of a convex metric space (X, d, W) and so T has a fixed point in $P_C(x_0) \subseteq C$. (see [8]). ■

Remarks 15.

1. From the proof of the above theorem we see that the set $P_C(x_0)$ is T-invariant whenever $x_0 \in X$ is a center for T.

2. For Banach spaces a result similar to Theorem **??** was proved in [3].

Corollary 16. Let C be a Chebyshev subset of a convex metric space (X, d, W). Then every continuous mapping $T : C \to C$ admitting a center in X has a fixed point.

Proof. Since C is a Chebyshev set, $P_C(x_0)$ is a singleton and so it is a compact convex subset of X and hence the result follows. ■

Since for an approximatively compact set C, the set $P_C(x)$ is non empty and compact for each $x \in X$ (see [9]), we have

Corollary 17. Let (X, d, W) be a convex metric space and C a nonempty approximatively compact convex set then every continuous mapping $T : C \to C$ admitting a center in X has a fixed point.

For mappings admitting strict center, we have

Theorem 18. Let C be a nonempty subset of a metric space (X, d) and $x_0 \in X$ be a strict center for $T : C \to C$. If $P_C(x_0)$ is nonempty then $P_C(x_0) \subseteq Fix(T)$.

Proof. On contrary, suppose $x \in P_C(x_0)$ is not a fixed point of T. Then $d(Tx, x_0) < d(x, x_0) = dist(x_0, C)$. But since $Tx \in C$, this inequality is not possible. ■

Remarks 19.

1. Above theorem shows that if T does not have a fixed point then $P_C(x_0) = \phi$.

2. For Banach spaces, Theorem 2 was proved by T. Dominguez Benavides et al. [3].

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