Global Journal of Mathematical Sciences: Theory and Practical.
ISSN 0974-3200 Volume 4, Number 2 (2012), pp. 133-141

http://www.irphouse.com

Operators and Dynamical System in Measurable Function Spaces

D. Senthilkumar and P. Chandra Kala*

Abstract

Let X be a Hausdorff topological space and let $B(E)$ be the Banach algebra of all bounded linear operators on a Banach space E. Let V be a system of weights on X. In this paper we make a study of dynamical system induced by multiplication operator and weighted composition operator on weighted spaces of measurable functions like $MV_0(X)$ (or $MV_0(X,E)$) and $MV_b(X)$ (or $MV_b(X,E)$) respectively.

Keywords: System of weights, measurable function, weighted composition operators, dynamical systems, seminorm, operator valued mapping.

2000 AMS Subject Classification: 47 B 37, 47 B 38, 47 B 07, 47 D 03, 30 H 05.

Introduction

L^p spaces are some of the most important spaces studied in mathematics because of our abundant usefulness and applications that run across all the branches of mathematics. This paper is a generalization of L^p spaces. Multiplication, composition operators and weighted composition operators have been appearing in a natural way on different spaces of continuous functions, analytic functions and cross-sections. For example [2, 3, 4, 6, 7] have shown that those operators on spaces of continuous and cross-sections. R.K.Singh and Manhas studied dynamical system induced by multiplication and composition operator on continuous and holomorphic function spaces [3, 5, 8, 9, 10]. We have organized this paper into four sections. In section 2, we characterized functions inducing multiplication operator on weighted locally convex spaces of measurable functions. In section 3 we obtained dynamical system induced by multiplication operators on weighted locally convex spaces of measurable functions. In the last section we wide up with dynamical system induced by weighted composition operators on weighted locally convex spaces of measurable functions.

Preliminaries

Let X be a Hausdorff topological space and $M(X,E)$ be the space of all measurable functions from X into E and $C(X,E)$ be the vector subspace of $M(X,E)$ consisting of the continuous functions f from X into E. Let V be a set of non-negative upper-semi continuous functions on X. If V is a set of weights on X such that given any $x \in X$, there is some $v \in V$ for which $v(x) > 0$. We write $V > 0$. A set V of weights on X is said to be directed upward provided for every pair $u_1, u_2 \in V$ and $\alpha > 0$ there exists $v \in V$ such that $\alpha u_i \le v$ (point wise on X) for $i = 1,2$. By a system of weights, we mean a set V of weights on X with additionally satisfies $V > 0$. Let $cs(E)$ be the set of all continuous functions from X into E. If V is a system of weights on X, then the pair (X,V) is called the weighted topological system. Associated with each weighted topological system (X,V), we have the weighted spaces of continuous E-valued functions defined as:

$$MV_o(X,E) = \{ f \in M(X,E) : vq(f) \text{ vanishes at } \infty \text{ on } X \text{ for each } v \in V, q \in cs(E)\}$$

$$MV_p(X,E) = \{ f \in M(X,E) : vq(f) \text{ in } L^p \text{ for all } v \in V, q \in cs(E)\}$$

$$MV_b(X,E) = \{ f \in M(X,E) : vq(f(X)) \text{ is bounded in } E \text{ for all } v \in V, q \in cs(E)\}$$

Let $v \in V$, $q \in cs(E)$ and $f \in M(X,E)$. If we define $\| f \|_{v,q} = (\int_X (v(x)q(f(x)))^p d\mu)^{\frac{1}{p}}$ *for all* $x \in X$, then $\|.\|_v$ can be regarded as a seminorm on either $MV_0(X,E)$, $MV_b(X,E)$ and the family $\left\{ \|.\|_{v,q} : v \in V, q \in cs(E) \right\}$ of seminorms defines a Hausdorff locally convex topology on each of these spaces. This topology will be denoted by ω_v and the vector spaces $MV_0(X,E)$ and $MV_b(X,E)$ endowed with ω_v are called the weighted locally convex space of vector-valued continuous functions. It has a basis of closed absolutely convex neighborhoods of the origin of the form, $B_{v,q} = \{ f \in MV_b(X,E) : \| f \|_{v,q} \le 1\}$

Also, $MV_0(X,E)$ is a closed subspace of $MV_b(X,E)$.\\

Let G be a topological group with e as identity, let X be a topological space and $\pi : G \times X \to X$ be a continuous map such that (i) $\pi(e,x) = x$ *for every* $x \in X$

(ii) $\pi(st,x) = \pi(s, \pi(t,x))$ *for every* $t,s \in G$ $x \in X$.

Then the triple (G,X,π) is called a transformation group, X is a state space. If $G = (R,+)$ the corresponding transformation group is called a dynamical system. The transformation group $(\mathbb{R}, X, \pi)$ is known as continuous dynamical system. If X is a Banach space and $\pi(t, \alpha x + \beta y) = \alpha\pi(t,x) + \beta\pi(t,y)$ *for* $t \in \mathbb{R}$, $\alpha, \beta \in \mathbb{C}$, $x,y \in X$ then $(\mathbb{R}^+, X, \pi)$ is called a linear dynamical system.

Functions inducing multiplication operators on weighted spaces of measurable functions

Theorem: 2.1

Let $\theta : X \to \mathbb{C}$ be a measurable function. Then $M_\theta : MV_0(X) \to MV_0(X)$ is a multiplication operator iff $V|\theta| \le V$.

Proof

First suppose $V|\theta| \le V$ .Then for every $v \in V$. Then for all $v \in V$, there exists $u \in V$ such that $v|\theta| \le u$ (point wise on X). We show that M_θ is a continuous linear operator on $MV_0(X)$. Clearly M_θ is linear on $MV_0(X)$. In order to prove the continuity of M_θ on $MV_0(X)$ it is enough to show that M_θ is continuous at origin. For this, suppose f_α be a net in $MV_0(X)$ such that $P_v f_\alpha \to 0$, for every $v \in V$.

Now

$$P_v(\theta f_\alpha) = (\int_X (v(x)q(\theta f_\alpha(x)))^p \, d\mu)^{\frac{1}{p}} \quad for\, all\, x \in X$$

$$\le (\int_X (u(x)q(\theta f_\alpha(x)))^p \, d\mu)^{\frac{1}{p}} = P_u(f_\alpha) \to 0.$$

This proves the continuity of M_θ at the origin and hence M_θ is continuous on $MV_0(X)$.

Conversely suppose M_θ is a continuous linear operator on $MV_0(X)$. We shall show that $V|\theta| \le V$. Let $v \in V$. Since M_θ is continuous origin, there exists $u \in V$ such that $M_\theta(B_u) \subseteq B_v$. We claim that $v|\theta| \le 2u$. Take $x_0 \in X$ and set $u(x_0) = \varepsilon$. In case $\varepsilon > 0$, $N = \{x \in X : u(x) < 2\varepsilon\}$ is an open neighborhood of x_0. Then there exists $f \in MV_0(X)$ such that $0 \le f \le 1, f(x_0) = 1$ and $f(X - N) = 0$. Let $g = (2\varepsilon)^{-1} f$. Then clearly $g \in B_u$. Since $M_\theta(B_u) \subseteq B_v$, we have $\theta g \in B_v$ and this yields that $v(x)|\theta(x)||g(x)| \le 1, \forall x \in X$. From this it follows that $v(x)|\theta(x)||f(x)| \le 2\varepsilon, \forall x \in X$. This implies that $v(x_0)|\theta(x_0)| \le 2u(x_0), \forall x \in X$. Now suppose $u(x_0) = 0$ and $v(x_0)|\theta(x_0)| > 0$. If we put $\varepsilon = v(x_0)|\theta(x_0)|$ its not greater than 2 and set $N = \{x \in X : u(x) < \varepsilon\}$ then N would be an open neighbourhood of x_0 and we could again find $f \in MV_0(X)$ such that $0 \le f \le 1, f(x_0) = 1$ and $f(X - N) = 0$. Now let $g = \varepsilon^{-1} f$. Then clearly $g \in B_u$ and $\theta g \in B_v$. Hence $v(x)|\theta(x)||g(x)| \le 1, \forall x \in X$. This implies that $v(x)|\theta(x)||f(x)| \le \varepsilon, \forall x \in X$. From this it follows that

$v(x_0)\left|\theta(x_0)\right| \le \frac{v(x_0)\left|\theta(x_0)\right|}{2}$ which is impossible. This proves our claim and hence the proof is complete.

Now we shall characterize multiplication operators on $MV_0(X,E)$ induced by scalar-valued and vector-valued functions.

Theorem: 2.2

Let $\theta : X \to \mathbb{C}$ be a measurable function. Then $M_\theta : MV_0(X,E) \to MV_0(X,E)$ is a multiplication operator iff $V\left|\theta\right| \le V$.

Proof

Similar to proof of therorem:2.1.

Theorem: 2.3

Let E be a (locally multiplicatively convex) lmc algebra with unit e and let $\psi : X \to E$ be a bounded measurable function. Then $M_\psi : MV_0(X,E) \to MV_0(X,E)$ is a multiplication operator if $V_p \circ \psi \le V, \forall p \in \mathcal{P}$.

Proof

Suppose $V_p \circ \psi \le V, \forall p \in \mathcal{P}$. Then $\forall v \in V, \exists u \in V \ni v_p \circ \psi \le u (\text{point wise on } X)$. We shall prove that the mapping $M_\psi : MV_0(X,E) \to M(X,E)$ defined by $M_\psi f = \psi f$, where the product is point wise continuous linear operator on $MV_0(X,E)$, We shall establish the continuity of M_ψ at the origin. For this, let $\{f_\alpha\}$ be a net in $MV_0(X,E)$ such that $\forall v \in V, q \in \mathcal{P}, \ P_{v,q}(f_\alpha) \to 0$

Then

$$P_{v,q}(\psi f_\alpha) = \left(\int (v(x) q(\psi(x) f_\alpha(x)))^p d\mu\right)^{\frac{1}{p}}$$

$$\le \left(\int (u(x) q(f_\alpha(x)))^p d\mu\right)^{\frac{1}{p}}$$

$$= P_{u,q}(f_\alpha) \to 0 .$$

This proves that M_ψ is continuous origin and hence a continuous linear operator on $MV_0(X,E)$.

Remark: 2.4

Note that if $\theta : X \to \mathbb{C}$ (or $\psi : X \to E$) is bounded measurable complex-valued(or vector-valued) function on X , then clearly M_θ (or M_ψ) is multiplication operator on $MV_0(X)$ (or $MV_0(X,E)$) for any system of weights V .

If V is system of weights generated by the characteristic functions of compact sets, then it turns out that every continuous map induces a multiplication operator on $MV_0(X)$ (or $MV_0(X,E)$) for any system of weights V.

Theorem: 2.5

Let X be a completely regular Haustorff space and let $V = \left\{\lambda_{\chi_K} : \lambda > 0 \, and \, K \subset X, K \, is \, compact\right\}$.

(i) Every bounded $\theta : X \to \mathbb{C}$ on $MV_0(X)$.

(ii) Every bounded $\psi : X \to E$ a lmc with jointly continuous multiplication induces a multiplication operator on M_ψ on $MV_0(X,E)$.

Proof

Similar proof of theorem: 2.3.

Corollary: 2.6

Let X have the discrete topology and $V = \left\{\lambda_{\chi_K} : \lambda \geq 0 \, and \, K \subset X, K \, is \, a \, finite \, set\right\}$. Then every function $\theta : X \to \mathbb{C}$ (or $\psi : X \to E$) induces a multiplication operator M_θ(or M_ψ) on $MV_0(X)$ (or $MV_0(X,E)$).

Example: 2.7

Let $\mathbb{R}^+$ be the set of positive real with usual topology and let $v : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $v(x) = \frac{1}{x}, \forall x \in \mathbb{R}^+$. Let $V = \{\lambda v : \lambda \geq 0\}$ and let $\theta : \mathbb{R}^+ \to \mathbb{C}$ be defined as $\theta(x) = x^2$. Then θ does not induce a multiplication operator on $MV_0(\mathbb{R}^+)$.

Dynamical system induced by multiplication operators on weighted locally convex space of measurable functions

Theorem: 3.1

Let U and V be arbitrary system of weights on G and let $\theta \in M(X)$ (or $\psi : X \to E$). Then $M_\theta : MU_b(X) \to MV_b(X)$ is a multiplication operator if $V|\theta| \leq U$.

Proof

To show that M_θ is a multiplication operator. It is enough to prove that is M_θ continuous at origin.Let $v \in V$ and B_v be a neighbourhood of the origin in $MV_b(X)$. Then by the given condition, there exists $u \in U$ such that $v|\theta| \leq u$. Now we claim that $M_\theta(B_u) \subseteq B_v$, where B_u is neighbourhood of the origin in $MU_b(X)$ (or $MU_b(X,E)$). Let $f \in B_u$. Then we have

$$\|M_\theta f\|_v = (\int_X (v(x)|\theta(x) f(x)|)^p d\mu)^{\frac{1}{p}}$$

This shows that $\theta f \in B_v$ and hence M_θ is a multiplication operator.

Corollary: 3.2
Every bounded measurable function $\theta : X \to \mathbb{C}$ induces a multiplication operator M_θ on $MV_b(X)$ (or $MV_b(X,E)$) for any system of weights V on X.

Proof
Since θ is bounded, $\exists m > 0 \ni |\theta(x)| \le m \; for\, all\, x \in X$. Let $v \in V$. Then we have $v(x)|\theta(x)| \le mv(x) \; for\, all\, x \in X$. That is, $\exists u \in V \ni v(x)|\theta(x)| \le u(x) \; for\, all\, x \in X$.

Hence by the above theorem M_θ is a multiplication operator on $MV_b(X)$ (or $MV_b(X,E)$).

Note 3.3
Let $g \in F_b(\mathbb{R})$. Define $\psi_t : \mathbb{R} \to B(T)$ as $\psi_t(\omega) = e^{tg(\omega)} \; for\, all\, t, \omega \in \mathbb{R}$.

Theorem: 3.4
Let $g \in F_b(\mathbb{R})$. For each $t \in \mathbb{R}$ and let $\nabla_g : \mathbb{R} \times MV_b(\mathbb{R},T) \to M(\mathbb{R},T)$ be the function defined by $\nabla_g(t,f) = M_{\psi_t} f \; for\, all\, t \in \mathbb{R}\, and\; f \in MV_b(\mathbb{R},T)$. Then ∇_g is a dynamical system on $MV_b(\mathbb{R},T)$.

Proof
Since M_{ψ_t} is a multiplication operator on $MV_b(\mathbb{R},T) \; for\, all\, t \in \mathbb{R}$. We can conclude that $\nabla_g(t,f) \in MV_b(\mathbb{R},T)$ whenever $t \in \mathbb{R}$ and $f \in MV_b(\mathbb{R},T)$. Thus ∇_g is a function from $\mathbb{R} \times MV_b(\mathbb{R},T) \to M(\mathbb{R},T)$. It can be easily seen that $\nabla_g(0,f) = f$ and $\nabla_g(t+s,f) = \nabla_g(t, \nabla_g(s,f))$. In order to show that ∇_g is a dynamical system on $MV_b(\mathbb{R},T)$.It is enough to show that ∇_g separately continuous map. Let us first prove the continuity of ∇_g in the first argument. Let $\{t_n \to t\}$. Then $|t_n - t| \to 0 \, as\, n \to \infty$. We shall show that $\nabla_g(t_n, f) \to \nabla_g(t,f)$ in $MV_b(\mathbb{R},T)$. Let $v \in V$ .Then

$$P(\nabla_g(t_n,f) - \nabla_g(t,f))_v = P(\psi_{t_n} f - \psi_t f)_v$$

$$= (\int_{\mathbb{R}} (v(\omega) q(\psi_{t_n}(\omega) f(\omega) - \psi_t(\omega) f(\omega)))^p d\mu)^{\frac{1}{p}} \; for\, all\, \omega \in \mathbb{R}$$

$$\leq (\int_{\mathbb{R}} (v(\omega)q(\psi_{t_n}(\omega))q(f(\omega)))^p d\mu)^{\frac{1}{p}} - (\int_{\mathbb{R}} (v(\omega)q(\psi_t(\omega)q(f(\omega)))^p d\mu)^{\frac{1}{p}} \; for\, all\, \omega \in \mathbb{R}$$

$$\leq (e^{|t_n| q(g(\omega))}((\int_{\mathbb{R}} (v(\omega)q(f(\omega)))^p d\mu)^{\frac{1}{p}} - (e^{|t| q(g(\omega))})(\int_{\mathbb{R}} (v(\omega)q(f(\omega)))^p d\mu)^{\frac{1}{p}}$$

$$\leq (e^{|t| q(g(\omega))})(e^{|t_n - t| q(g(\omega))} - I)(\int_{\mathbb{R}} (v(\omega)q(f(\omega)))^p d\mu)^{\frac{1}{p}} \to 0\, as\, |t_n - t| \to 0.$$

Let f_α be a net in $MV_b(\mathbb{R},T)$ such that $f_\alpha \to f$ in $MV_b(\mathbb{R},T)$. Then $q(f_\alpha \to f)_v \to 0\; for\, all\, v \in V$. We shall show that $\nabla_g(t, f_\alpha) \to \nabla_g(t, f)$ in $MV_b(\mathbb{R},T)$.

For this, let $v \in V$. Then

$$P(\nabla_g(t, f_\alpha) - \nabla_g(t, f))_v = P(\psi_t(\omega) f_\alpha(\omega) - \psi_t(\omega) f(\omega) : \omega \in \mathbb{R})_v$$

$$\leq (e^{|t| q(g(\omega))}) \int_{\mathbb{R}} (v(\omega)q(f_\alpha(\omega) - f(\omega)))^p d\mu)^{\frac{1}{p}}$$

$$\leq (e^{|t| q(g(\omega))}) P(f_\alpha - f)_v \to 0\, as\; f_\alpha \to f.$$

This proves the continuity of ∇_g is a (linear) dynamical system on the weighted space $MV_b(\mathbb{R},T)$.

Dynamical system and weighted composition operator

Theorem: 4.1 [2]

Let E be a locally convex Hausdorff space such that each convergent net in E is bounded. Let $\psi \in M(X, B(E))$ and $T \in M(X, X)$. Then $W_{\psi,T}$ is a weighted composition operator on $MV_b(X,E)$ iff for every $v \in V$ and $p \in cs(E)$, $\exists u \in V$ and $q \in cs(E)$ such that $v(x)p(\psi_x(y)) \leq u(T(x))q(y) \forall x \in X\; and\; y \in E$.

Remark: 4.2

Let $B(E)$ be the Banach algebra of all bounded linear operators on E. Then an operator-valued map $\psi_t : X \to B(E)$ defined by $\psi_t(x) = e^{tg(x)}\; for\, all\, t \in \mathbb{R}\, and\; x \in X$, where $g \in M_b(X, B(E))$ and $\|g\|_\infty = \sup\{\|g(x)\| : x \in X\}$. Also $T_t : X \to X$ is defined by $T_t(x) = t + x$ the self-map. Then the weighted composition operator induced by ψ_t and T_t on the spaces of $MV_0(X,E)$ and $MV_b(X,E)$.

Theorem: 4.3

Let V be an arbitrary system of weights on X . Let $\nabla : \mathbb{R} \times MV_b(X,E) \to M(X,E)$ be the function defined by $\nabla(t, f) = W_{\psi_t, T_t} f\; for\, all\, t \in \mathbb{R}\, and\; f \in MV_b(X,E)$. Then ∇ is a linear dynamical system if for every $v \in V$ and $p \in cs(E)$ $\exists u \in V$ and $q \in cs(E)$ such that $v(x)p(\psi_x(y)) \leq u(T(x))q(y) \forall x \in X \text{ and } y \in E$.

Proof

For every $t \in \mathbb{R}, W_{\psi_t, T_t}$ is a weighted composition operator on $MV_b(X,E)$. Thus it follows that, $\nabla(t, f) \in MV_b(X,E)$ for all $t \in \mathbb{R} \text{ and } f \in MV_b(X,E)$ Clearly, ∇ is linear and

$$\nabla(0, f)(x) = W_{\psi_0, T_0} f(x) \text{ for all } x \in X$$
$$= \psi_0(x) f(0+x) = f(x) \text{ for all } x \in X.$$

Therefore $\nabla(0, f) = f$.

Also $\nabla(t+s, f) = \nabla(t, \nabla(s, f))$.

Next, to show that ∇ is linear dynamical system, it sufficient to show that ∇ is jointly continuous map[1]. Let $t_n \to t \in \mathbb{R}$. Then $t_n - t \to 0$ as $n \to \infty$. The remaining proof is similar to theorem:3.4.

Acknowledgement

The work was supported by Rajiv Gandhi National Fellowship under the University Grants Commission Fellowship and a bursary awarded to P. Chandra Kala.

References

[1] Chernoff.P and Marsden, On continuity and smoothness of group actions, Bull.Amer.Math.Soc., 76(1970), 1044-1049.

[2] J.S.Manhas, Weighted composition operators on weighted locally convex spaces of analytic function, Southeast Asian Bull.Math.29(2005), no.1, 127-140.

[3] J.H.Shpiro, Composition operators and classical function theory, Universitext:Tracts in Mathematics, Springer, New York, NY, USA, 1993.

[4] R.K. Singh and J.S.Manhas, Composition operators on function spaces, North Holland Mathematics Studies, 1993.

[5] R.K. Singh and J.S.Manhas, Multiplication operators and Dynamical systems, J.Austral.Math.Soc.(series A)53(1992), 92-102.

[6] R.K. Singh and J.S.Manhas, Composition operator on weighted spaces of vector-valued continuous functions, Acta.Sci.Math, 1998.

[7] R.K. Singh and D.C.Kumar, Weighted composition operators on functional Hilbert spaces, Bull.Aust.Math.Soc., 31(1985), 117-126.

[8] R.K. Singh and J.S.Manhas, Multiplication operators and Dynamical systems on weighted spaces of cross-section, Proc.AMS, vol 119, No 2, 1993.

[9] R.K. Singh and J.S.Manhas, Weigted composition operators on weighted spaces of cross-sections, Acta scientiarum Mathematicarum, vol 59, No 3-4, 1994.
[10] J.S.Manhas, Composition operators and Multiplication operator on weighted spaces of analytic functions, Research article, International Journal of Mathematics and Mathematical sciences, vol 2007.