Operators and Dynamical System in Measurable Function Spaces

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Abstract

Let $X$ be a Hausdorff topological space and let $B(E)$ be the Banach algebra of all bounded linear operators on a Banach space $E$. Let $V$ be a system of weights on $X$. In this paper we make a study of dynamical system induced by multiplication operator and weighted composition operator on weighted spaces of measurable functions like $MV_0(X)$ (or $MV_0(X,E)$) and $MV_0(X)$ (or $MV_0(X,E)$) respectively.

Keywords: System of weights, measurable function, weighted composition operators, dynamical systems, seminorm, operator valued mapping.

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Introduction

$L^p$ spaces are some of the most important spaces studied in mathematics because of our abundant usefulness and applications that run across all the branches of mathematics. This paper is a generalization of $L^p$ spaces. Multiplication, composition operators and weighted composition operators have been appearing in a natural way on different spaces of continuous functions, analytic functions and cross-sections. For example [2, 3, 4, 6, 7] have shown that those operators on spaces of continuous and cross-sections. R.K.Singh and Manhas studied dynamical system induced by multiplication and composition operator on continuous and holomorphic function spaces [3, 5, 8, 9, 10]. We have organized this paper into four sections. In section 2, we characterized functions inducing multiplication operator on weighted locally convex spaces of measurable functions. In section 3 we obtained dynamical system induced by multiplication operators on weighted locally convex spaces of measurable functions. In the last section we wide up with dynamical system induced by weighted composition operators on weighted locally convex spaces of measurable functions.
Preliminaries

Let $X$ be a Hausdorff topological space and $M(X,E)$ be the space of all measurable functions from $X$ into $E$ and $C(X,E)$ be the vector subspace of $M(X,E)$ consisting of the continuous functions $f$ from $X$ into $E$. Let $V$ be a set of non-negative upper-semi continuous functions on $X$. If $V$ is a set of weights on $X$ such that given any $x \in X$, there is some $v \in V$ for which $v(x) > 0$. We write $V > 0$. A set $V$ of weights on $X$ is said to be directed upward provided for every pair $u, \mu \in V$ and $\alpha > 0$ there exists $v \in V$ such that $au \leq v$ (point wise on $X$) for $i = 1, 2$. By a system of weights, we mean a set $V$ of weights on $X$ with additionally satisfies $V > 0$. Let $cs(E)$ be the set of all continuous functions from $X$ into $E$. If $V$ is a system of weights on $X$, then the pair $(X, V)$ is called the weighted topological system. Associated with each weighted topological system $(X, V)$, we have the weighted spaces of continuous $E$-valued functions defined as:

$MV_0(X, E) = \{f \in M(X, E) : vq(f) \text{ vanishes at } \infty \text{ on } X \text{ for each } v \in V, q \in cs(E)\}$

$MV_p(X, E) = \{f \in M(X, E) : vq(f) \in L^p \text{ for all } v \in V, q \in cs(E)\}$

$MV_b(X, E) = \{f \in M(X, E) : vq(f(X)) \text{ is bounded in } E \text{ for all } v \in V, q \in cs(E)\}$

Let $v \in V$, $q \in cs(E)$ and $f \in M(X, E)$. If we define

$\|f\|_{v,q} = \left(\int_X (v(x)q(f(x)))^\frac{1}{p}d\mu\right)^p$ for all $x \in X$, then $\|\|_v$ can be regarded as a seminorm on either $MV_0(X, E)$, $MV_p(X, E)$ and the family $\left\{\|\|_{v,q} : v \in V, q \in cs(E)\right\}$ of seminorms defines a Hausdorff locally convex topology on each of these spaces. This topology will be denoted by $\omega_v$ and the vector spaces $MV_0(X, E)$ and $MV_p(X, E)$ endowed with $\omega_v$ are called the weighted locally convex space of vector-valued continuous functions. It has a basis of closed absolutely convex neighborhoods of the form,

$B_{v,q} = \{f \in MV_0(X, E) : \|f\|_{v,q} \leq 1\}$

Also, $MV_0(X, E)$ is a closed subspace of $MV_p(X, E)$. \|

Let $G$ be a topological group with $e$ as identity, let $X$ be a topological space and $\pi : G \times X \to X$ be a continuous map such that (i) $\pi(e, x) = x$ for every $x \in X$

(ii) $\pi(st, x) = \pi(s, \pi(t, x))$ for every $t, s \in G$, $x \in X$.

Then the triple $(G, X, \pi)$ is called a transformation group, $X$ is a state space. If $G = (R, +)$ the corresponding transformation group is called a dynamical system. The transformation group $(R, X, \pi)$ is known as continuous dynamical system. If $X$ is a Banach space and $\pi(t, ax + by) = a\pi(t, x) + b\pi(t, y)$ for $t \in R$, $a, b \in C$, $x, y \in X$ then $(R^+, X, \pi)$ is called a linear dynamical system.
Functions inducing multiplication operators on weighted spaces of measurable functions

**Theorem: 2.1**

Let $\theta : X \rightarrow \mathbb{C}$ be a measurable function. Then $M_\theta : MV_0(X) \rightarrow MV_0(X)$ is a multiplication operator iff $V|\theta| \leq V$.

**Proof**

First suppose $V|\theta| \leq V$. Then for every $v \in V$. Then for all $v \in V$, there exists $u \in V$ such that $v|\theta| \leq u$ (point wise on $X$). We show that $M_\theta$ is a continuous linear operator on $MV_0(X)$. Clearly $M_\theta$ is linear on $MV_0(X)$. In order to prove the continuity of $M_\theta$ on $MV_0(X)$ it is enough to show that $M_\theta$ is continuous at origin. For this, suppose $f_\alpha$ be a net in $MV_0(X)$ such that $P(f_\alpha) \rightarrow 0$, for every $v \in V$.

Now

$$P_\mu(\theta f_\alpha) = \left( \int_X (v(x)q(\theta f_\alpha(x)))^p d\mu \right)^{1/p} \text{ for all } x \in X$$

$$\leq \left( \int_X (u(x)q(\theta f_\alpha(x)))^p d\mu \right)^{1/p} = P_\mu(f_\alpha) \rightarrow 0.$$  

This proves the continuity of $M_\theta$ at the origin and hence $M_\theta$ is continuous on $MV_0(X)$.

Conversely suppose $M_\theta$ is a continuous linear operator on $MV_0(X)$. We shall show that $V|\theta| \leq V$. Let $v \in V$. Since $M_\theta$ is continuous origin, there exists $u \in V$ such that $M_\theta(B_u) \subseteq B_v$. We claim that $v|\theta| \leq 2u$. Take $x_0 \in X$ and set $u(x_0) = \varepsilon$. In case $\varepsilon > 0$, $N = \{x \in X : u(x) < 2\varepsilon\}$ is an open neighborhood of $x_0$. Then there exists $f \in MV_0(X)$ such that $0 \leq f \leq 1, f(x_0) = 1$ and $f(X - N) = 0$. Let $g = (2\varepsilon)^{-1}f$. Then clearly $g \in B_{\varepsilon}$. Since $M_\theta(B_{\varepsilon}) \subseteq B_v$, we have $\theta g \in B_v$ and this yields that $v(x)|\theta(x)|g(x)| \leq 1, \forall x \in X$. From this it follows that $v(x)|\theta(x)||f(x)| \leq 2\varepsilon, \forall x \in X$. This implies that $v(x_0)|\theta(x_0)| \leq 2u(x_0), \forall x \in X$. Now suppose $u(x_0) = 0$ and $v(x_0)|\theta(x_0)| > 0$. If we put $\varepsilon = v(x_0)|\theta(x_0)|$ its not greater than $2$ and set $N = \{x \in X : u(x) < \varepsilon\}$ then $N$ would be an open neighbourhood of $x_0$ and we could again find $f \in MV_0(X)$ such that $0 \leq f \leq 1, f(x_0) = 1$ and $f(X - N) = 0$. Now let $g = \varepsilon^{-1}f$. Then clearly $g \in B_{\varepsilon}$ and $\theta g \in B_v$. Hence $v(x)|\theta(x)||g(x)| \leq 1, \forall x \in X$. This implies that $v(x)|\theta(x)||f(x)| \leq \varepsilon, \forall x \in X$. From this it follows that
Now we shall characterize multiplication operators on $MV_0(X,E)$ induced by scalar-valued and vector-valued functions.

**Theorem: 2.2**

Let $\theta : X \to \mathbb{C}$ be a measurable function. Then $M_{\theta} : MV_0(X,E) \to MV_0(X,E)$ is a multiplication operator iff $V|\theta| \leq V$.

**Proof**

Similar to proof of theorem:2.1.

**Theorem: 2.3**

Let $E$ be a (locally multiplicatively convex) lmc algebra with unit $e$ and let $\psi : X \to E$ be a bounded measurable function. Then $M_{\psi} : MV_0(X,E) \to MV_0(X,E)$ is a multiplication operator if $V_{\psi} \phi \psi \leq V, \forall \phi \in \mathcal{P}$.

**Proof**

Suppose $V_{\psi} \phi \psi \leq V, \forall \phi \in \mathcal{P}$. Then $\forall x \in V, \exists u \in V \forall \phi \psi \leq u$ (point wise on $X$). We shall prove that the mapping $M_{\psi} : MV_0(X,E) \to M(X,E)$ defined by $M_{\psi} f = \psi f$, where the product is point wise continuous linear operator on $MV_0(X,E)$. We shall establish the continuity of $M_{\psi}$ at the origin. For this, let $\{f_a\}$ be a net in $MV_0(X,E)$ such that $\forall x \in V, q \in \mathcal{P}, P_{v,q}(f_a) \to 0$

Then

$$P_{v,q}(\psi f_a) = \left( \int (v(x)q(\psi(x)f_a(x)))^p d\mu \right)^{\frac{1}{p}}$$

$$\leq \left( \int (u(x)q(f_a(x)))^p d\mu \right)^{\frac{1}{p}}$$

$$= P_{u,q}(f_a) \to 0 .$$

This proves that $M_{\psi}$ is continuous origin and hence a continuous linear operator on $MV_0(X,E)$.

**Remark: 2.4**

Note that if $\theta : X \to \mathbb{C}$ (or $\psi : X \to E$) is bounded measurable complex-valued(or vector-valued) function on $X$, then clearly $M_{\theta}$ (or $M_{\psi}$) is multiplication operator on $MV_0(X)$ (or $MV_0(X,E)$) for any system of weights $V$. 

$$v(x_0)|\theta(x_0)| \leq \frac{v(x_0)|\theta(x_0)|}{2} \text{ which is impossible. This proves our claim and hence the proof is complete.}$$
If $V$ is a system of weights generated by the characteristic functions of compact sets, then it turns out that every continuous map induces a multiplication operator on $MV_0(X)$ (or $MV_0(X, E)$) for any system of weights $V$.

**Theorem: 2.5**

Let $X$ be a completely regular Hausdorff space and let $V = \{\lambda_{\mathcal{K}} : \lambda > 0$ and $\mathcal{K} \subset X, \mathcal{K}$ is compact\}.

(i) Every bounded $\theta : X \to \mathbb{C}$ on $MV_0(X)$.

(ii) Every bounded $\psi : X \to E$ a lmc with jointly continuous multiplication induces a multiplication operator on $M_\psi$ on $MV_0(X, E)$.

**Proof**

Similar proof of theorem: 2.3.

**Corollary: 2.6**

Let $X$ have the discrete topology and $V = \{\lambda_{\mathcal{K}} : \lambda \geq 0$ and $\mathcal{K} \subset X, \mathcal{K}$ is a finite set\}.

Then every function $\theta : X \to \mathbb{C}$ (or $\psi : X \to E$) induces a multiplication operator $M_\theta$ (or $M_\psi$) on $MV_0(X)$ (or $MV_0(X, E)$).

**Example: 2.7**

Let $\mathbb{R}^+$ be the set of positive real with usual topology and let $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\nu(x) = \frac{1}{x}, \forall x \in \mathbb{R}^+$. Let $V = \{\lambda_\nu : \lambda \geq 0\}$ and let $\theta : \mathbb{R}^+ \to \mathbb{C}$ be defined as $\theta(x) = x^2$. Then $\theta$ does not induce a multiplication operator on $MV_0(\mathbb{R}^+)$.  

Dynamical system induced by multiplication operators on weighted locally convex space of measurable functions

**Theorem: 3.1**

Let $U$ and $V$ be arbitrary system of weights on $G$ and let $\theta \in M(X)$ (or $\psi : X \to E$). Then $M_\theta : MU_b(X) \to MV_b(X)$ is a multiplication operator if $\nu \theta \leq U$.

**Proof**

To show that $M_\theta$ is a multiplication operator. It is enough to prove that is $M_\theta$ continuous at origin.Let $\nu \in V$ and $B_\nu$ be a neighbourhood of the origin in $MU_b(X)$. Then by the given condition, there exists $u \in U$ such that $\nu \theta \leq u$. Now we claim that $M_\theta(B_\nu) \subseteq B_\nu$, where $B_\nu$ is neighbourhood of the origin in $MU_b(X)$ (or $MU_b(X, E)$). Let $f \in B_\nu$. Then we have
\[ \|M_\theta f\| = \left( \int_X (\nu(x)|\theta(x)f(x)|^p \, d\mu)^{\frac{1}{p}} \right) \]

This shows that \( \theta f \in B_v \) and hence \( M_\theta \) is a multiplication operator.

**Corollary: 3.2**
Every bounded measurable function \( \theta : X \to \mathbb{C} \) induces a multiplication operator \( M_\theta \) on \( MV_b(X) \) (or \( MV_b(X,E) \)) for any system of weights \( V \) on \( X \).

**Proof**
Since \( \theta \) is bounded, \( \exists m > 0 \ \forall \theta(x) \leq m \) for all \( x \in X \). Let \( \nu \in V \). Then we have \( \nu(x)|\theta(x)| \leq m\nu(x) \) for all \( x \in X \). That is, \( \exists u \in V \ \forall \nu(x)|\theta(x)| \leq u(x) \) for all \( x \in X \).

Hence by the above theorem \( M_\theta \) is a multiplication operator on \( MV_b(X) \) (or \( MV_b(X,E) \)).

**Note 3.3**
Let \( g \in F_b(\mathbb{R}) \). Define \( \psi_t : \mathbb{R} \to B(T) \) as \( \psi_t(\omega) = e^{gt(\omega)} \) for all \( t, \omega \in \mathbb{R} \).

**Theorem: 3.4**
Let \( g \in F_b(\mathbb{R}) \). For each \( t \in \mathbb{R} \) and let \( \nabla_g : \mathbb{R} \times MV_b(\mathbb{R},T) \to M(\mathbb{R},T) \) be the function defined by \( \nabla_g(t,f) = M_{\psi_t}f \) for all \( t \in \mathbb{R} \) and \( f \in MV_b(\mathbb{R},T) \). Then \( \nabla_g \) is a dynamical system on \( MV_b(\mathbb{R},T) \).

**Proof**
Since \( M_{\psi_t} \) is a multiplication operator on \( MV_b(\mathbb{R},T) \) for all \( t \in \mathbb{R} \). We can conclude that \( \nabla_g(t,f) \in MV_b(\mathbb{R},T) \) whenever \( t \in \mathbb{R} \) and \( f \in MV_b(\mathbb{R},T) \). Thus \( \nabla_g \) is a function from \( \mathbb{R} \times MV_b(\mathbb{R},T) \to M(\mathbb{R},T) \). It can be easily seen that \( \nabla_g(0,f) = f \) and \( \nabla_g(t+s,f) = \nabla_g(t,\nabla_g(s,f)) \). In order to show that \( \nabla_g \) is a dynamical system on \( MV_b(\mathbb{R},T) \). It is enough to show that \( \nabla_g \) separately continuous map. Let us first prove the continuity of \( \nabla_g \) in the first argument. Let \( \{t_n \to t\} \). Then \( |t_n - t| \to 0 as n \to \infty \). We shall show that \( \nabla_g(t_n,f) \to \nabla_g(t,f) \) in \( MV_b(\mathbb{R},T) \). Let \( \nu \in V \). Then

\[ P(\nabla_g(t_n,f) - \nabla_g(t,f))_{\nu} = P(\psi_{t_n}f - \psi_t f)_{\nu} \]

\[ = \left( \int_\mathbb{R} (\nu(\omega)q(\psi_{t_n}(\omega)f(\omega) - \psi_t(\omega)f(\omega)))^p \, d\mu)^{\frac{1}{p}} \text{ for all } \omega \in \mathbb{R} \]
\[
\int_{\mathbb{R}} (v(\omega)q(\psi(\omega))q(f(\omega)))^p d\mu \leq (e^{H_{\psi,g}(\omega)}) ((\int_{\mathbb{R}} (v(\omega)q(f(\omega)))^p d\mu)^q - (e^{H_{\psi,g}(\omega)}) (\int_{\mathbb{R}} (v(\omega)q(f(\omega)))^p d\mu)^{\frac{1}{q}})
\]

Let \( f_\alpha \) be a net in \( MV_b(\mathbb{R},T) \) such that \( f_\alpha \to f \) in \( MV_b(\mathbb{R},T) \). Then \( q(f_\alpha \to f)_v \to 0 \) for all \( v \in V \). We shall show that \( \nabla_g(t, f_\alpha) \to \nabla_g(t, f) \) in \( MV_b(\mathbb{R},T) \).

For this, let \( v \in V \). Then
\[
P(\nabla_g(t, f_\alpha) - \nabla_g(t, f))_v = P(\psi_t(\omega)f_\alpha(\omega) - \psi_t(\omega)f(\omega) : \omega \in \mathbb{R})_v
\]

This proves the continuity of \( \nabla_g \) is a (linear) dynamical system on the weighted space \( MV_b(\mathbb{R},T) \).

**Dynamical system and weighted composition operator**

**Theorem: 4.1** [2]

Let \( E \) be a locally convex Hausdorff space such that each convergent net in \( E \) is bounded. Let \( \psi \in M(X,E) \) and \( T \in M(X,X) \). Then \( W_{\psi,T} \) is a weighted composition operator on \( MV_b(X,E) \) iff for every \( v \in V \) and \( p \in cs(E) \), \( \exists u \in V \) and \( q \in cs(E) \) such that \( v(x)p(\psi_u(\omega)) \leq u(T(x))q(y) \forall x \in X \) and \( y \in E \).

**Remark: 4.2**

Let \( B(E) \) be the Banach algebra of all bounded linear operators on \( E \). Then an operator-valued map \( \psi_t : X \to B(E) \) defined by \( \psi_t(x) = e^{gt(x)} \) for all \( t \in \mathbb{R} \) and \( x \in X \), where \( g \in M_b(X,B(E)) \) and \( \|g\|_\infty = \sup \{\|g(x)\| : x \in X\} \). Also \( T_t : X \to X \) is defined by \( T_t(x) = t + x \) the self-map. Then the weighted composition operator induced by \( \psi_t \) and \( T_t \) on the spaces of \( MV_b(X,E) \) and \( MV_b(X,E) \).

**Theorem: 4.3**

Let \( V \) be an arbitrary system of weights on \( X \). Let \( \nabla : \mathbb{R} \times MV_b(X,E) \to M(X,E) \) be the function defined by \( \nabla(t, f) = W_{\psi_t,T} f \) for all \( t \in \mathbb{R} \) and \( f \in MV_b(X,E) \). Then \( \nabla \) is
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a linear dynamical system if for every \( v \in V \) and \( p \in \text{cs}(E) \) \( \exists u \in V \) and \( q \in \text{cs}(E) \) such that \( \psi(x)p(\psi(x)(y)) \leq u(T(x))q(y) \forall x \in X \) and \( y \in E \).

**Proof**

For every \( t \in \mathbb{R}, W_{v,x,t} \) is a weighted composition operator on \( MV_{b}(X,E) \). Thus it follows that, \( \nabla(t,f) \in MV_{b}(X,E) \) for all \( t \in \mathbb{R} \) and \( f \in MV_{b}(X,E) \) Clearly, \( \nabla \) is linear and

\[
\nabla(0,f)(x) = W_{\psi_0,\tau_0}f(x) \text{ for all } x \in X \\
= \psi_0(x)f(0 + x) = f(x) \text{ for all } x \in X .
\]

Therefore \( \nabla(0,f) = f \).

Also \( \nabla(t + s,f) = \nabla(t,\nabla(s,f)) \).

Next, to show that \( \nabla \) is linear dynamical system, it sufficient to show that \( \nabla \) is jointly continuous map[1]. Let \( t_n \to t \in \mathbb{R} \). Then \( t_n - t \to 0 \) as \( n \to \infty \). The remaining proof is similar to theorem:3.4.

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**References**


