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A Remark on Riemann Integrable Functions Defined on an Interval

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Abstract

For f a Riemann integrable function on [a,b], we prove that the continuity of f at a point c is a necessary condition for the indefinite integral $\int_{a}^{x} f$ to be differentiable at c. This means that the map $F \mapsto \frac{d}{dx}F$ carries the indefinite integral of each function $f \in R[a,b]\}$ to a function in C[a,b] or rather, if the derivative of the indefinite integral of f exists at a point c, then it is precisely f(c).

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Introduction

Concerning the definitions for the Lebesgue and the Riemann integral ([2], [1]) of a bounded function $f:[a,b]\subset R\to R$ it is well known that the derivative $\frac{d}{dx}\int_{a}^{x} f=f(x)$ a.e. and, further, if f is Riemann integrable and continuous at the point c, then $\frac{d}{dx}\int_{a}^{x} f(c)=f(c)$. We consider the upper limit $\lim_{x\to c} f(x)$ and the analogue lower limit $\lim_{x\to c} f(x)$. We see easily in The Result, paragraph 3., that $\frac{d}{dx}\int_{a}^{x} f(t)dt=\lim_{x\to c} f(x)$ and analogously $\frac{d}{dx}\int_{a}^{x} f(t)dt=\lim_{x\to c} f(x)$ where the integrals are respectively the upper integral and the lower integral Darboux integrals. Thus the derivative of the indefinite Riemann integral must coincide with $\lim_{x\to c} f(x)=\lim_{x\to c} f(x)$ if it exists and, consequently, f is continuous at c and the result in the Abstract follows.

Preliminaries

We consider a bounde function $f:[a,b]\subset R\to R$.

For such a bounded function f clearly that $\lim_{x\to c} f(x)=\inf_{\varepsilon>0}\{\sup\{f(c+h):|h|<\varepsilon\}\}$ is a real number and, interchanging inf↔sup, also the lower limit $\lim_{x\to c} f(x)$ exists at each point $c\in(a,b)$, with the understanding fot h if c is a endpoint.

Considering the oscillation $\omega(f,c)=\inf\{\omega(f,I(c,\varepsilon)\cap[a,b]):\varepsilon>0\}$ where $\omega(f,A)=\sup\{|f(x)-f(y)|:x,y\in A\cap[a,b]\}$, we have the

Lemma: The function f is Riemann integrable if and only if $\omega(f,x)$ is integrable and $\int_{a}^{b}\omega(f,x)dx=0$.

Proof: This follows clearly from Riemann's condition as in the proof of Theorem 7.19. in [1], pp. 153-4 as wished.

The Result

Theorem: For $f:[a,b]\subset R\to R$ a bounded Riemann integrable function, if the derivative $\frac{d}{dx}$ of the indefinite integral $\int_{a}^{x} f$ exists at a point c, then f is continuous at c.

Proof: Let $L=\lim_{x\to c} f(x)$. We have that for each $\delta>0$, there is a partition $a=x_0<x_1<\ldots<x_{N}=b$ of $[a,b]$ such that $\int_{a}^{b}(L-f(x))dx\leq \sum_{k=1}^{N}\sup\{L-f(x):x\in(x_{k-1},x_{k})\}(x_{k}-x_{k-1})+\delta\leq\sum_{k=1}^{N}\omega(f,(x_{k-1},x_{k})\cap[a,b])(x_{k}-x_{k-1})+\delta$ and it follows that $\int_{c}^{c+h}(L-f)(x)dx=\int_{c}^{c+h}(L-f(x))dx=0$. Therefore $|\lim_{h\to 0}\int_{c}^{c+h}f(x)dx/h-L|=\lim_{h\to 0}\int_{c}^{c+h}(L-f(x))/h=\lim 0=0$. Analogously, we see that $l=\lim_{h\to 0} f(x)=\lim_{h\to 0}\int_{c}^{c+h}f(x)dx/h$ hence $L=l$ and the theorem is proved.

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