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# General System for $\phi$-Strongly Accretive Nonlinear Variational Inequalities in $q$-Uniformly Smooth Banach Spaces

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**Abstract**

We prove that recent results of Chen and Xing [ Int. J. Comtemp. Math. Science, Vol. 2, 2007, no.23, 1121-1127 ] concerning the general system for strongly accretive nonlinear variational inequalities in $q-$uniformly smooth Banach spaces based on the convergence of sunny nonexpansive retraction projection methods can be extended to much more general class of $\phi-$ strongly accretive mappings. The results presented in this paper extend and improve the results of Chen and Xing (2007).

**Keyword and Phrases**: General system for $\phi-$strongly accretive nonlinear variational inequalities, sunny nonexpansive mapping, $q-$uniformly smooth.

## Introduction

Projection methods have played a significant role in the numerical resolution of variational inequalities in Hilbert spaces. And Verma [1] introduces the general two-step model for projection methods, which reduces to the-step model applied in [2] and then applies it to the approximately in a Hilbert space setting.

It is the aim of this paper to improve the result of Chen and Xing [8] in $q-$ uniformly smooth Banach spaces. In order to overcome the difficulties caused by the lack of projections, we will restrict our investigation in smooth Banach space because in such a space, the fixed point set of a nonexpansive mapping is a sunny nonexpansive retract (see definition in section 2). Since a sunny nonexpansive retraction in terms of a duality mappings enjoys some of the nice properties that projection in Hilbert space has, we are able to establish the main result in a smooth

Banach space setting. Let $X$ be a real smooth Banach space with dual $X^*$, we denote by $J$ the normalized duality mapping from $X$ to $2^{X^*}$. It is well known that if $X$ is normalized duality by $J$. Let K be a nonempty closed convex subset of $X$ and let $A: K \to K$ be any mapping on $K$. We consider system of two nonlinear variational inequality (abbreviated as SNVI) problems as follows: to find elements

$x^*, y^* \in K$ such that

$$\left\langle \rho A(y^*) + x^* - y^*, j(x - x^*) \right\rangle \geq 0, \forall x, y \in K \text{ and for } \rho > 0 \tag{1.1}$$

$$\left\langle \eta A(x^*) + y^* - x^*, j(x - y^*) \right\rangle \geq 0, \forall x, y \in K \text{ and for } \eta > 0 \tag{1.2}$$

The SNVI problem (1.1) and (1.2) is equivalent to the following sunny nonexpansive retraction projection formulas

$$x^* = P_k\left[y^* - \rho A(y^*)\right] \text{ for } \rho > 0$$

$$y^* = P_k\left[x^* - \eta A(x^*)\right] \text{ for } \eta > 0$$

Where $P_k$ is the sunny nonexpansive retraction projection from $X$ onto $K$. Next we consider two special cases of SNVI problem (1.1) and (1.2)

1. If $\eta = 0$, then the SNVI problem (1.1) and (1.2) reduces to the following nonlinear variational inequality (NVI) problem: to find an $x^* \in K$ such that

$$\left\langle A(x^*), j(x - x^*) \right\rangle \geq 0, \forall x \in K \tag{1.3}$$

2. If $K$ is a closed convex cone of $X$, then the SNVI problem (1.1) and (1.2) is equivalent to the following system of nonlinear complementarity (SNC) problems to find $x^*, y^* \in K$ such that

$A(x^*), A(y^*) \in K^*$ and

$$\left\langle \rho A(y^*) + x^* - y^*, j(x^*) \right\rangle = 0, \text{ for } \rho > 0 \tag{1.4}$$

$$\left\langle \eta A(x^*) + y^* - x^*, j(y^*) \right\rangle = 0, \text{ for } \eta > 0 \tag{1.5}$$

Where $K^*$ is the polar cone to $K$ defined by

$$K^* = \left\{ f \in X : \left\langle f, j(x) \right\rangle \geq 0, \forall x \in K \right\}.$$

## Preliminaries

Throughout this paper, we always let $X$ be a real Banach space with the dual space $X^*$. The generalized duality mappings

$J_q(x): X \to 2^{X^*}$ is defined by

$$J_q(x) = \left\{ f \in X^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1} \right\} \forall x \in X$$

Where $q > 1$ is a constant. In particular, $J_2 = J$ is the usual normalized duality mapping. It is known that $J_q = \|x\|^{q-2} J$ for all $x \in X$, and $J_q(x)$ is single-valued if $X^*$ is strictly convex. In the sequel, unless otherwise specified, we always suppose that $X$ is a real Banach space such that $J_q$ is single-valued. We denote the single-valued generalized duality by $j_q$. If $X$ is a Hilbert space, then $J$ becomes the identity mapping of $X$.

The modules of smoothness of $X$ is the function $\rho x : [0,+\infty) \to [0,+\infty)$ defined by

$$\rho x(t) = \sup\left\{ \frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| = 1, \|y\| = t, x \in X, y \in X \right\}, t > 0$$

If there exists constant $c > 0$ and a real number $1 < q < \infty$, such that $\rho x(t) \le ct^q$, then $X$ is said to be uniformly smooth. A Banach space $X$ is called uniformly smooth if $\lim_{t \to \infty} \rho x(t)/t = 0$.

In the sequel, we will give some definitions.

**Definition 2.1.** Let $A : X \to X$ be a single-valued operation, then the operator $A$ is said to be

Accretive if $\langle Ax - Ay, j_q(x-y) \rangle \ge 0, \forall x, y \in X$

Strictly Accretive if $\langle Ax - Ay, j_q(x-y) \rangle \ge 0, \forall x, y \in X$ and the inequality holds if only $y = x$;

Strongly Accretive if there exists a constant $L > 0$, such that

$$\langle Ax - Ay, j_q(x-y) \rangle \ge L\|x-y\|^q, \forall x, y \in X$$

$\phi -$Strongly Accretive if there exists strictly increasing function $\phi$: $[0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle Ax - Ay, j_q(x-y) \rangle \ge \phi\|x-y\|\|x-y\|, \forall x, y \in X$$

$$\ge \frac{\phi\|x-y\|}{(1 + \phi\|x-y\| + \|x+y\|)}\|x-y\|^q = \tau(x,y)\|x-y\|^q$$

where

$$\tau(x,y)=\frac{\phi\|x-y\|}{(1+\phi\|x-y\|+\|x+y\|)}\in[0,1),\forall x,y\in E$$

Lipschitz continuous if there exists a constant $L>0$, such that

$$\|Ax-Ay\|\geq L\|x-y\|^{q},\forall x,y\in X$$

**Definition 2.2.** Let $C$ and $K$ be nonempty subset of a Banach space $X$ such that $C$ is nonempty closed convex and $K\subset C,$ then a mapping $P_k:C\to K$ is called

1. Retraction from $C$ onto $K$ if $P_k x=x,\forall x\in K.$
2. Sunny if $P_k(P_k x+t(x-P_k x))=P_k x,\forall x\in C.$

Whenever $(P_k x+t(x-P_k x))\in C$ and $t>0$.

3. A sunny nonexpansive retraction if $P_k$ is sunny, nonexpansive and a retraction of $C$ onto $K$. The following lemma is well known (see reference [3,4]).

**Lemma 2.1.** Let $C$ be a nonempty convert subset of a smooth Banach space $X$, $K\subset C,$ $J:X\to X^*$ the (normalized) duality mapping of $X$, and $P_k:C\to K$ a retraction. Then the following are equivalent:

1. $\langle x-P_k x, j(y-P_k x)\rangle\leq 0,\forall x\in C$ and $y\in K$;
2. $P_k$ is both sunny and nonexpansive.

In order to prove our main result, we need the following lemmas.

**Lemma 2.2 ([5])** Let $\{\lambda_n\}$be a sequence in $[0,1)$such that $\lim_{n\to\infty}\lambda_n=0$ then

$$\sum_{n=1}^{\infty}\lambda_n=\infty\Leftrightarrow\prod_{n=1}^{\infty}(1-\lambda_n)=0.$$

**Lemma 2.3. ([6)]** let $X$ be a real uniformly smooth Banach space. Then, $X$ is $q-$ uniformly smooth if and only if there exist a constant $c>0$, such that for all $x,y\in X$

$$\|x+y\|^q\leq\|x\|^q+q\langle y, j_q(x)\rangle+c\|y\|^q$$

## Algorithms

In this section, we deal with an introduction in general two-step models for sunny nonexpansive retraction projection and its special forms that can be applied to the convergence analysis for sunny nonexpansive retraction projection in the context of the approximation solvability of the SNVI problem (1.1) and (1.2).

**Algorithm 3.1** For arbitrarily chosen initial points $x_1, y_1 \in K$, computing the sequences $\{x_n\}, \{y_n\}$ such that

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_k[y_n - \rho A(y_n)]$$
$$y_n = (1-\beta_n)x_n + \beta_n P_k[x_n - \eta A(x_n)]$$

Where $P_k$ is the sunny nonexpansive mapping of $X$ onto $K$, $\rho$ and $\eta > 0$ constants and $\{\alpha_n\}, \{\beta_n\}$ are sequences in [0.1].

For $\{\beta_n\} = 1$ in algorithm 3.1, we get

**Algorithm 3.2** For arbitrary chosen initial points $x_1, y_1 \in K$, computing the sequences $\{x_n\}, \{y_n\}$ such that

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_k[y_n - \rho A(y_n)]$$
$$y_n = \beta_n P_k[x_n - \eta A(x_n)]$$

where $P_k$ is the sunny nonexpansive mapping of $X$ onto $K$, $\rho$ and $\eta > 0$ constants and $\{\alpha_n\}$ is a sequence in [0.1].

For $\eta = 0$ in algorithm 3.1, we get

**Algorithm 3.3** For an arbitrary chosen initial point $x_1 \in K$, computing the sequence $\{x_n\}$ such that

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_k[x_n - \rho A(x_n)]$$

Where $P_k$ is the sunny nonexpansive mapping of $X$ onto $K$, $\rho > 0$ is a constant and $\{\alpha_n\}$ is a sequence in [0.1].

## Main Result

We now present on Algorithm 3.1, the approximation-solvability of the SNVI problem (1.1) and (1.2) involving a mapping $A : K \to X$ which is $\phi-$strongly accretive and Lipschits continuous with a function $r(x, y)$ and constant $L$, respectively, in a $q-$uniformly smooth Banach space setting.

**Theorem 4.1** Let $X$ be a real $q-$uniformly Smooth Banach Spaces, and $K$ be a nonempty closed convex subset of $X$ and $T$: $K \to K$ be $\phi-$strongly accretive and $L-$ Lipschitz continuous mapping. $P_K : X \to K$ sunny nonexpansive retraction mapping. Suppose that $x^*, y^* \in K$ from a solution of the SNVI problem (1.1) and (1.2), the sequence $\{x_n\},\{y_n\}$ are generated by the Algorithm 3.1 and $\alpha_n, \beta_n \in [0,1]$, $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$. Then, sequences $\{x_n\},\{y_n\}$, respectively, converges to $x^*$ and $y^*$ for $0 < \rho^{q-1} < \dfrac{r(x,y)q}{cL^q}$, $0 < \eta^{q-1} < \dfrac{r(x,y)q}{cL^q}$.

**Proof** Since $x^*$ and $y^*$ form a solution to the SNVI problem (1.1) and (1.2), it follows that

$$x^* = P_k[y^* - \rho A(y^*)]$$
$$y^* = P_k[x^* - \rho A(x^*)]$$

Applying Algorithm 3.1, we have

$$\|x_{n+1} - x^*\| = \|(1-\alpha_n)x_n + \alpha_n P_k[y_n - \rho A(y_n)] - (1-\alpha_n)x^* + \alpha_n P_k[y^* - \rho A(y^*)]\|$$
$$\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|P_k[y_n - \rho A(y_n)] - P_k[y^* - \rho A(y^*)]\|$$
$$\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|y_n - y^* - \rho[A(y_n) - \rho A(y^*)]\| \qquad (4.1)$$

Since A is $\phi$ – Strongly accretive and $L$ – Lipschitz continuous, we have

$$\|y_n - y^* - \rho[A(y_n) - \rho A(y^*)]\|^q \le \|y_n - y^*\|^q - q\rho$$
$$\langle A(y_n) - \rho A(y^*), j_q(y_n - y^*)\rangle + c\rho^q \|A(y_n) - \rho A(y^*)\|^q$$
$$\le \|y_n - y^*\|^q - q\rho\tau(y_n, y^*)\|y_n - y^*\|^q + c\rho^q L^q \|y_n - y^*\|^q$$
$$= [1 - q\rho\tau(y_n, y^*) + c\rho^q L^q]\|y_n - y^*\|^q$$

Since $D = \sup\{\|A(x_n) - x^*\| + \|A(y_n) - x^*\| : n \ge 0\} + \|x_0 + x^*\|$

Then by induction, we obtain that $\|x_n - x^*\| \le D, \forall n \ge 0$ and let liminf $\|x_n - x^*\| = \delta \ge 0, \forall n \ge 0$, then there exists a positive number $N_0$ such that $\|x_n - x^*\| \ge \dfrac{\delta}{2}, \forall n \ge 0$.

Hence, it follows from (4.3) that

$$\|y_n - y^* - \rho[A(y_n) - \rho A(y^*)]\|^q$$
$$\le \left[1 - \frac{\varphi\|y_n - y^*\|}{\left(1 + \varphi\|y_n - y^*\| + \|y_n + y^*\|\right)} q\rho + c\rho^q L^q\right]\|y_n - y^*\|^q$$

$$\left\| y_n - y^* - \rho\left[A(y_n) - \rho A(y^*)\right]\right\|$$

$$\leq \left[1 - \frac{\varphi\left(\frac{\delta}{2}\right)}{\left(1+\varphi(D)+D\right)} q\rho + c\rho^q L^q\right]^{1/q} \left\| y_n - y^* \right\| \tag{4.2}$$

Substitute (4.2) into (4.1) to obtain

$$\left\| x_{n+1} - x^* \right\| \leq (1-\alpha_n)\left\| x_n - x^* \right\| + \alpha_n$$

$$\left[1 - \frac{\varphi\left(\frac{\delta}{2}\right)}{\left(1+\varphi(D)+D\right)} q\rho + c\rho^q L^q\right]^{1/q} \left\| y_n - y^* \right\| \tag{4.3}$$

$$\left\| x_{n+1} - x^* \right\| \leq (1-\alpha_n)\left\| x_n - x^* \right\| + \alpha_n \theta \left\| y_n - y^* \right\|$$

$$= (1-\alpha_n)\left\| x_n - x^* \right\| + \alpha_n \left\| y_n - y^* \right\| \tag{4.4}$$

where

$$0 < \theta = \left[1 - \frac{\phi\left(\frac{\delta}{2}\right)}{\left(1+\phi(D)+D\right)} q\rho + c\rho^q L^q\right]^{1/q} < 1$$

$$\left\| y_n - y^* \right\| = \left\| (1-\beta_n)x_n + \beta_n P_k\left[x_n - \eta A(x_n)\right] - (1-\alpha_n)x^* + \alpha_n P_k\left[x^* - \rho A(x^*)\right]\right\|$$

$$\leq (1-\beta_n)\left\| x_n - x^* \right\| + \beta_n \left\| P_k\left[x_n - \eta A(x_n)\right] - P_k\left[x^* - \rho A(x^*)\right]\right\|$$

$$\leq (1-\beta_n)\left\| x_n - x^* \right\| + \beta_n \left\| x_n - x^* - \eta\left[A(x_n) - A(x^*)\right]\right\| \tag{4.6}$$

Since A is $\phi-$Strongly accretive and $L-$Lipschitz continuous, we have

$$\left\| x_n - x^* - \eta\left[A(x_n) - A(x^*)\right]\right\|^q \leq \left\| x_n - x^* \right\|^q - q\eta$$

$$\left\langle A(x_n) - A(x^*), j_q(x_n - x^*)\right\rangle + c\eta^q \left\| A(x_n) - A(x^*)\right\|^q$$

$$\leq \left\| x_n - x^* \right\|^q - q\eta\tau(x_n, x^*)\left\| x_n - x^* \right\|^q + c\eta^q L^q \left\| x_n - x^* \right\|^q$$

$$= \left[ 1 - \frac{\phi \left\| x_n - x^* \right\|}{\left(1 + \phi \left\| x_n - x^* \right\| + \left\| x_n + x^* \right\|\right)} q\eta + c\eta^q L^q \right] \left\| x_n - x^* \right\|^q$$

Thus,

$$\left\| x_n - x^* - \eta \left[ A(x_n) - A(x^*) \right] \right\|^q$$

$$\leq \left[ 1 - \frac{\varphi \left\| x_n - x^* \right\|}{\left(1 + \varphi \left\| x_n - x^* \right\| + \left\| x_n + x^* \right\|\right)} q\eta + c\eta^q L^q \right] \left\| x_n - x^* \right\|^q$$

$$\left\| x_n - x^* - \eta \left[ A(x_n) - A(x^*) \right] \right\| \leq \left[ 1 - \frac{\phi\left(\frac{\delta}{2}\right)}{\left(1 + \phi(D) + D\right)} q\eta + c\eta^q L^q \right]^{1/q} \left\| x_n - x^* \right\| \quad (4.7)$$

Substitute (4.7) into (4.6) to obtain

$$\left\| y_n - x^* \right\| \leq (1 - \beta_n) \left\| x_n - x^* \right\| + \beta_n \left[ 1 - \frac{\phi\left(\frac{\delta}{2}\right)}{\left(1 + \phi(D) + D\right)} q\eta + c\eta^q L^q \right]^{1/q} \left\| x_n - x^* \right\|$$

$$\left\| y_n - x^* \right\| \leq (1 - \beta_n) \left\| x_n - x^* \right\| + \beta_n \sigma \left\| x_n - x^* \right\| \quad (4.8)$$

where

$$0 < \sigma = \left[ 1 - \frac{\phi\left(\frac{\delta}{2}\right)}{\left(1 + \phi(D) + D\right)} q\eta + c\eta^q L^q \right]^{1/q} < 1$$

Substitute (4.8) into (4.4) to obtain

$$\left\| x_{n+1} - x^* \right\| \leq (1 - \alpha_n) \left\| x_n - x^* \right\| + \alpha_n \left[ (1 - \beta_n) \left\| x_n - x^* \right\| + \beta_n \sigma \left\| x_n - x^* \right\| \right]$$

$$= (1 - \alpha_n) \left\| x_n - x^* \right\| + \alpha_n (1 - \beta_n) \left\| x_n - x^* \right\| + \alpha_n \beta_n \sigma \left\| x_n - x^* \right\|$$

$$= \left[ (1 - \alpha_n) + \alpha_n (1 - \beta_n) + \alpha_n \beta_n \sigma \right] \left\| x_n - x^* \right\|$$

$$= \left[ (1 - \alpha_n) + \alpha_n (1 - \beta_n) + \alpha_n \beta_n \sigma \right] \left\| x_n - x^* \right\|$$

$$= [1-\alpha_n+\alpha_n-\alpha_n\beta_n+\alpha_n\beta_n\sigma]\|x_n-x^*\|$$
$$= [1-(1-\sigma)\alpha_n\beta_n]\|x_n-x^*\|$$
$$\leq \prod_{n=1}^{n}[1-(1-\sigma)\alpha_n\beta_n]\|x_1-x^*\| \tag{4.9}$$

Since $0<\sigma<1$ and $\sum_{n=1}^{\infty}\alpha_n\beta_n=\infty$, thus by Lemma 2.3,

$$\prod_{n=1}^{\infty}[1-(1-\sigma)\alpha_n\beta_n]=0$$

Hence, the sequence $\{x_n\}$ converges to $x^*$ by (4.9) and $\{y_n\}$ converges to $y^*$ by (4.6)

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