On Vertex-Cover Polynomial on Some Standard Graphs

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Abstract

The vertex cover Polynomial of a graph $G$ of order $n$ has been already introduced in [3]. It is defined as the polynomial, $C(G, x) = \sum_{i=1}^{\beta(G)} c(G, i)x^i$, where $c(G, i)$ is the number of vertex covering sets of $G$ of size $i$ and $\beta(G)$ is the covering number of $G$. We obtain some properties of $C(G, x)$ and its coefficients for some standard graphs. Also, we compute the vertex cover polynomials for $K_n - \{v\}$, the product graph $K_m \times K_n$, the net graph, the Peterson graph, the cubic graph and $K_n \circ K_1$.

Keywords: Vertex covering set, vertex covering number, vertex cover polynomial.

Introduction

Let $G = (V, E)$ be a simple graph. For any vertex $v \in V$, the open neighborhood of $v$ is the set $N(v) = \{u \in V : uv \in E\}$ and the closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of $S$ is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of $S$ is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a vertex covering of $G$ if every edge $uv \in E$ is adjacent to at least one vertex in $S$. The vertex covering number $\beta(G)$ is the minimum cardinality of the vertex covering sets in $G$. A vertex covering...
set with cardinality $\beta(G)$ is called a $\beta$-set. Let $C(G, i)$ be the family of vertex covering sets of $G$ with cardinality $i$ and let $c(G, i) = |C(G, i)|$. The polynomial, $C(G, x) = \sum_{i=\beta(G)}^{v(G)} c(G, i) x^i$ is defined as the vertex cover polynomial of $G$. In [3], many properties of the vertex cover polynomials have been studied and derived the vertex cover polynomials for some standard graphs. In this paper, we find the expression for vertex cover polynomial of disjoint union of graphs. Also, we find vertex cover polynomials for some standard graphs such as the Peterson graph, Net graph etc.

The number of edges incident to the vertex $v$ of a graph $G$ is called the degree of the vertex $v$ in $G$. It is denoted by $\deg(v)$. $\delta(G)$ and $\Delta(G)$ are the minimum and maximum of the degree of all the vertices in $G$ respectively. The composition $G_1[G_2]$ as having $V = V_1 \times V_2$ and $u = (u_1, v_1)$ and $v = (u_2, v_2)$ are adjacent if $u_1$ is adjacent to $u_2$ in $G_1$ or $u_1 = u_2$ and $v_1$ is adjacent to $v_2$ in $G_2$. The product $G_1 \times G_2$ as having $V = V_1 \times V_2$ and $u = (u_1, v_1)$ and $v = (u_2, v_2)$ are adjacent if $u_1 = u_2$ and $v_1$ is adjacent to $v_2$ in $G_2$ or $v_1 = v_2$ and $u_1$ is adjacent to $u_2$ in $G_1$.

**Vertex Cover Polynomials**

**Definition 2.1:** Let $C(G, i)$ be the family of all vertex covering sets of $G$ with cardinality $i$ and let $c(G, i) = |C(G, i)|$. The vertex cover polynomial of $G$ is defined as

$$C(G, x) = \sum_{i=\beta(G)}^{v(G)} c(G, i) x^i.$$ 

In [3] the vertex cover polynomial of $K_n$ is obtained as $C(K_n, x) = nx^{n-1} + x^n$.

**Theorem 2.2** Let $K_n$ be the complete graph with $n$ vertices.

Then, $C(K_n - \{v\}, x) = \frac{1}{n} \frac{d}{dx} [C(K_n, x)]$.

**Proof:** As $K_n$ is complete with $n$ vertices, $K_n - \{v\}$ is complete with $n - 1$ vertices. We have,

$$C(K_n, x) = x^n \left(1 + \frac{n}{x}\right)$$  \hspace{1cm} (i)

and

$$C(K_n - \{v\}, x) = x^{n-1} \left(1 + \frac{n - 1}{x}\right)$$  \hspace{1cm} (ii)
Differentiating (i) we get,
\[
\frac{d}{dx} [C(K_n, x)] = \frac{d}{dx} \left[ x^n \left( 1 + \frac{n}{x} \right) \right]
\]
\[
= n x^{n-1} \left( 1 + \frac{n-1}{x} \right)
\]

Therefore,
\[
\frac{1}{n} \frac{d}{dx} [C(K_n, x)] = x^n \left( 1 + \frac{n-1}{x} \right)
\]
\[
= C(K_n, \{v\}, x) \text{ [by (ii)]}
\]

Therefore, \( C(K_n - \{v\}, x) = \frac{1}{n} \frac{d}{dx} [C(K_n, x)] \)

**Remark: 2.3**

If \( G_1 \) and \( G_2 \) are any two graphs with the same degree sequence, then \( C(G_1, x) \) and \( C(G_2, x) \) need not be the same.

(eg)

\( C(G_1, x) = x^2 + 4x^3 + 9x^4 + 6x^5 + x^6 \); \( C(G_2, x) = 4x^3 + 9x^4 + 6x^5 + x^6 \)

therefore, \( C(G_1, x) \neq C(G_2, x) \)

**Definition: 2.4.** Let \( G_1 \) and \( G_2 \) be any two graphs. If \( C(G_1, x) = C(G_2, x) \), then \( G_1 \) and \( G_2 \) are said to be \( C \)-equivalence graphs.

**Note: 2.5**

Two non isomorphic graphs can be \( C \)-equivalence. (eg)
Theorem: 2.6
Let $K_m$ and $K_n$ be the complete graphs with $m$, $n$ vertices respectively. Then the vertex cover polynomial of the cross product $K_m \times K_n$ is
\[
C(K_m \times K_n, x) = nC_0 [mC_{m-1} \cdot (m-1) C_{m-2} \ldots (n \text{ terms})] x^{n(m-1)} + nC_1 [mC_{m-1} \cdot (m-1) C_{m-2} \ldots (n-1) \text{ terms}] x^{n(m-1)+1} + nC_2 [mC_{m-1} \cdot (m-1) C_{m-2} \ldots (n-2) \text{ terms}] x^{n(m-1)+2} + \ldots + nC_{n-1} (mC_{m-1}) x^{n-1} + nC_n x^n
\]

Proof: $K_m \times K_n$ is the graph with vertex set $V = \{v_{ij} = (u_i, v_j) / u_i \in K_m$ and $v_j \in K_n, i = 1 \ldots m, j = 1 \ldots n\}$ and $|V| = mn$. Now the vertices of $G = K_m \times K_n$ can be arranged in the form of a matrix of order $m \times n$.

Now, each column in the graph given in figure 3 represents the graph $K_m$. We need any $m-1$ elements to cover all edges of Column-1. In a similar way, we have to select $m-1$ elements in each column to cover all edges of $G$. Now the vertices of $G$ in the first column are denoted by $v_{11}, v_{21}, v_{m1}$. The minimum vertex covering set of $G$ is any $(m-1)$ elements in the first column among the vertices, $v_{i1}, i = 1 \ldots m$. It can be selected in $m$ ways. Suppose the element $v_{k1}$ is not selected in the first column, then the element $v_{k2}$ should be selected in the second column in the corresponding row, and $(m-2)$ elements have to be selected from the remaining $(m-1)$ elements in the second column other than $v_{k2}$. Similarly, suppose the element $v_{t2}$ was not selected in the second column, then the element $v_{t3}$ should be selected in the third column.
Thus two elements are fixed in the third column. The \( m - 3 \) elements in column 3 have to be selected from \( m - 2 \) elements and so on.

Therefore, \( c(G, n (m - 1)) = nC_0 [mC_{m-1} + (m - 1) C_{m-2} + \ldots + (n \text{ terms})] \)

Similarly, \( c(G, n (m - 1) +1) = nC_1 [mC_{m-1} + (m-1) C_{m-2} + \ldots + (n - 1) \text{ terms}] \)

\[ c((G, n (m - 1) + 2) = nC_2 [mC_{m-1} + (m - 1) C_{m-2} + \ldots \text{ (n-2 terms)}], \ldots \]

\[ c(G, mn - 1) = nC_{n-1} [mC_{m-1}] \quad \& \quad c(G, mn) = 1 \]

Therefore,

\[ C(K_m \times K_n, x) = nC_0 [mC_{m-1} \cdot (m - 1) C_{m-2} \cdot \ldots \text{ (n \text{ terms})}] \cdot x^n (m - 1) \]

\[ + nC_1 [mC_{m-1} \cdot (m - 1) C_{m-2} \cdot \ldots (n - 1) \text{ terms}] \cdot x^{n(m - 1) + 1} \]

\[ + nC_2 [mC_{m-1} \cdot (m - 1) C_{m-2} \cdot \ldots (n - 2) \text{ terms}] \cdot x^{n(m - 1) + 2} \]

\[ + \ldots + nC_{n-1} [mC_{m-1}] x^n m - 1 + nC_n x^n m. \]

Example: 2.7

\[ \begin{array}{l}
    G_1 = K_4 \\
    (\text{Figure 4 a}) \\
    G_2 = K_3 \\
    (\text{Figure 4 b}) \\
    K_4 \times K_3 \\
    \text{(Figure 4 c)}
\end{array} \]
Here \( m = 4 \) and \( n = 3 \)

\[
C(G, x) = 3C_0 [4C_3 \cdot 3C_2 \cdot 2C_1] x^9 + 3C_1 [4C_3 \cdot 3C_2] x^{10} + 3C_2 [4C_3] x^{11} + 3C_3 x^{12} \\
C(G, x) = 24x^9 + 36x^{10} + 12x^{11} + x^{12}
\]

**Corollary 2.8**: For the complete graphs, \( K_m \) and \( K_n \)

\[
C(K_m [K_n], x) = x^{mn} \left[ 1 + \frac{mn}{x} \right]
\]

**Proof**: Composition of \( K_m \) and \( K_n \) is a complete graph with \( mn \) vertices.

Then \( C(G, x) = mn x^{mn} - 1 + x^{mn} = x^{mn} \left[ 1 + \frac{mn}{x} \right], \) where \( G = K_m \circ [K_n], \) since for a complete graph of order \( n, \) the polynomial is \( nx^{n-1} + x^n. \)

**Note: 2.9**

If \( G_1 \) and \( G_2 \) are complete, then \( C(G_1 [G_2], x) = C(G_2 [G_1], x) \)

**Theorem 2.10**: If \( G_1 \) and \( G_2 \) are any two disjoint graphs, then

\[
C(G_1 \lor G_2, x) = C(G_1, x) \cdot C(G_2, x).
\]

**Proof**

Since \( G_1 \) and \( G_2 \) are any two graphs with \( G_1 \cap G_2 = \varnothing \). The selection of vertex covering set of \( G_1 \) does not affect the terms of \( G_2 \).

Therefore, \( C(G_1 \lor G_2, x) = C(G_1, x) \cdot C(G_2, x). \) (eg)

\[
\begin{align*}
\text{(Figure 5 a)} & \quad \text{(Figure 5 b)} & \quad \text{(Figure 5 c)} \\
\text{G}_1 & \quad \text{G}_2 & \quad \text{G}_1 \circ \text{G}_2
\end{align*}
\]

\[
C(G_1, x) = x + 3x^2 + x^3; \quad C(G_2, x) = 2x + x^2 \\
C(G_1 \lor G_2, x) = x^2 + 7x^3 + 11x^4 + 6x^5 + x^6 = C(G_1, x) \cdot C(G_2, x)
\]

\( \square \)
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Note

If \( G_1 \cap G_2 \neq \emptyset \), then
\[
C(G_1 \lor G_2, x) \neq C(G_1, x) \cdot C(G_2, x)
\]

Example 2.11

\[C(G_2, x) = x + 3x^2 + x^3; \quad C(G_2, x) = 2x^2 + x^2\]

\[C(G_1 \otimes G_2, x) = x + 3x^2 + 4x^3 + x^4\] (i)

\[C(G_1, x \tilde{\cup} G_2, x) = 2x^2 + x^3 + 5x^4 + x^5\] (ii)

From (i) & (ii) \( C(G_1 \mkern-2mu \sqcup \mkern-2mu G_2, x) \neq C(G_1, x) \cdot C(G_2, x) \)

Theorem 2.12

The vertex cover polynomial of any net graph \( G \) with \( n \times n \) vertices \( (n > 2) \) is

i) \[
C(G, x) = 2x^k \cdot [1 + x]^{k+1} + 4x^k + 1 \cdot [1 + x]^{k+2} + 2 + 4(n - 2) \cdot x^k + 3 \cdot [1 + x]^{k+3} - 3 \]

\[+ (n - 2)^2 \cdot x^k + 3 \cdot [1 + x]^{k+4} - x^{2k} - 1 \cdot [n^2 + 2n + x]\]

if \( n \) is even, where \( k = \left\lfloor \frac{n^2}{2} \right\rfloor \)

ii) \[
C(G, x) = x^k \cdot [1 + x]^{k+1} + 4 \cdot (p + 2) \cdot x^k + 2 \cdot [1 + x]^{k+2} + 2 \cdot (4p + q) \cdot x^k + 3 \cdot [1 + x]^{k+3} - 3 + 4 \cdot k + 1 \cdot [1 + x]^{k+4} + (q + 1) \cdot x^k + 4 \cdot [1 + x]^{k+5} - x^{2k} \cdot (8p + 2q + 9 + x)\]

for any odd \( n \), where \( k = \left\lfloor \frac{n^2}{2} \right\rfloor \); \( T = n - 2 \); \( p = \left\lfloor \frac{T}{2} \right\rfloor \); \( q = \left\lfloor \frac{T^2}{2} \right\rfloor \)

Proof:

(i) The net graph with \( n \times n \) \((n \text{ is even})\) can be converted into bipartite graph as follows. A graph of \( 4 \times 4 \) vertices and its corresponding bipartite is shown below in (figure 7 & 8).
Let \( S_1 = \{v_1, v_2, \ldots, v_k\} \) and \\
\( S_2 = \{u_1, u_2, \ldots, u_k\} \) \\
where \( k = \left\lfloor \frac{n^2}{2} \right\rfloor \).

Therefore, \( |S_1| = |S_2| = k \).

In \( S_1 \) and \( S_2 \), two vertices are of degree 2; \( 2(n-2) \) vertices are of degree 3 and \( \frac{(n-2)^2}{2} \) vertices of degree 4. The minimum vertex covering set with cardinality \( k \) is \( S_1 \) or \( S_2 \). Therefore, \( c(G, k) = 2 \). The vertex covering set with cardinality \( k + 1 \) are any one of the following form:

\[
\begin{align*}
S_1 & \chi \{u_i\} , S_2 \chi \{v_i\} ; \\
S_1 - \{v_i\} & \chi \ N (v_i) ; \ S_2 - \{u_i\} \chi \ N (u_i) / d (u_i) = d (v_i) = 2
\end{align*}
\]

Therefore, \( c (G, k + 1) = 2 (kC_1) + 4 \)

The sets having cardinality \( k + 2 \) are any one of the following form:

\[
\begin{align*}
S_1 \chi \{u_i, u_j\} , S_2 \chi \{v_i, v_j\}/u_i, u_j S_2 & \chi \ N (v_i) & \chi \ N (v_j) ; \\
S_1 - \{v_i\} & \chi \ N (v_i) / u_i \not\subset N (v_i) \& \deg (v_i) = 2, \\
S_2 - \{u_i\} & \chi \ N (u_i) / v_i \not\subset N (u_i) \& \deg (u_i) = 2, \\
S_1 - \{v_i\} & \chi \ N (v_i) / \deg (v_i) = 3, \\
S_2 - \{u_i\} & \chi \ N (u_i) / \deg (u_i) = 3
\end{align*}
\]

Therefore, \( c (G, k+2) = 2 (kC_2) + 4 ((k-2) C_1) + 4 (n - 2) \)

The covering sets with \( k + 3 \) elements are any one of the following form
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\[ S_1 \not\subset \{u_i, u_j, u_k\}, S_2 \not\subset \{v_i, v_j, v_k\}, \]
\[ S_1 - \{v_i\} \not\subset N(v_i) \not\subset \{u_i, u_j\} \not\subset N(v_i) & \deg(v_i) = 2, \]
\[ S_2 - \{u_i\} \not\subset N(u_i) \not\subset \{v_i, v_j\} \not\subset N(u_i) & \deg(u_i) = 2, \]
\[ S_1 - \{v_i\} \not\subset N(v_i) \not\subset \{u_i, u_j\} \not\subset N(v_i) & \deg(v_i) = 3, \]
\[ S_2 - \{u_i\} \not\subset N(u_i) \not\subset \{v_i, v_j\} \not\subset N(u_i) & \deg(u_i) = 3, \]
\[ S_1 - \{v_i\} \not\subset N(v_i) & \deg(v_i) = 4, \]
\[ S_2 - \{u_i\} \not\subset N(u_i) & \deg(u_i) = 4. \]

Therefore, \( c(G, k + 3) = 2(kC_3) + 4((k - 2)C_2) + 4(n - 2)((k - 3)C_1) + (n - 2)^2 \)

In a similar way,
\[ c(G, k + 4) = 2(kC_4) + 4((k - 2)C_3) + 4(n - 2) + ((k - 3)C_2) + (n - 2)^2((k - 4)C_1) \]
\[ c(G, k + 5) = 2(kC_5) + 4((k - 2)C_4) + 4((n - 2)((k - 3)C_3) + (n - 2)^2((k - 4)C_2) \]
\[ c(G, 2k - 3) = 2(kC_{k - 3}) + 4((k - 2)C_{k - 4}) + 4(n - 2)((k - 3)C_{k - 5} + (n - 2)^2(k - 4)C_{k - 6}) \]
\[ c(G, 2k - 2) = 2(kC_{k - 2}) + 4((k - 2)C_{k - 3}) + 4(n - 2)((k - 3)C_{k - 4}) + (n - 2)^2((k - 4)C_{k - 5}) \]
\[ c(G, 2k - 1) = 2(kC_{k - 1}) \quad \text{and} \quad c(G, 2k) = 1 \]

Therefore, \( C(G, x) = 2x^k + 2[kC_1x + x^k + 1 + C_2x^k + 2 + C_3x^k + 3 + \ldots + C_{k - 2}x^{2k - 2} + C_{k - 1}x^{2k - 1} + C_kx^{2k}] - x^{2k} + 4[x^{k + 1} + (k - 2)C_1x^{k + 2} + (k - 2)C_2x^{k + 3} + \ldots + (k - 2)C_{k - 4}x^{k + 3} + (k - 3)C_{k - 5}x^{k + 4} + \ldots + (k - 3)C_{k - 5}x^{2k - 3} + (k - 3)C_{k - 4}x^{2k - 2} + (n - 2)^2[x^{k + 3} + (k - 4)C_1x^{k + 4} + (k - 4)C_2x^{k + 5} + \ldots + (k - 4)C_{k - 5}x^{2k - 3} + (k - 4)C_{k - 6}x^{2k - 2} + (k - 3)C_{k - 7}x^{2k - 3} + (k - 3)C_{k - 6}x^{2k - 4} + (k - 2)C_{k - 7}x^{2k - 5} + (k - 4)C_kx^{k - 4} + (k - 6)C_{k - 5}x^{k - 5} + (k - 7)C_{k - 6}x^{k - 6} + (k - 8)C_{k - 7}x^{k - 7}] \]
\[
= 2x^k [1 + x]^k - x^{2k} + 4x^k + 1 \quad [1 + x]^k - 2 - 4x^{2k} - 1 + 4(n - 2)x^k + 2 [1 + x]^k - 3 \\
- 4(n - 2)x^{2k - 1} + (n - 2)^2x^{k + 1} [1 + x]^k - 2 + 4(n - 2)x^k + 2 [1 + x]^k - 3 + (n - 2)^2x^k + 3 [1 + x]^k - 4 - x^{2k - 1} [n^2 + 2n + x].
\]

(ii) The net graph with \(n \times n\) vertices (\(n\) is odd) can be converted as a bipartite graph as follows.

For example, a graph of \(5 \times 5\) vertices and its corresponding bigraph is shown below in figure 9 & 10.

Let \(S_1 = \{(v_1, v_2, v_3, \ldots, v_k + 1)\}\).
$S_2 = \{(u_1, u_2, \ldots, u_k) \text{ where } k = \left\lfloor \frac{n^2}{2} \right\rfloor$}

Let $T = n - 2$; $p = \left\lfloor \frac{T}{2} \right\rfloor$ and $q = \left\lfloor \frac{T^2}{2} \right\rfloor$.

In $S_1$, 4 vertices are of degree 2, $4p$ vertices are of degree 3 and $q + 1$ vertices are of degree 4.

In $S_2$, no vertices are of degree 2. $4(p + 1)$ vertices are of degree 3 and $q$ vertices are of degree 4. Also, $|S_1| = k + 1$; $|S_2| = k$.

Clearly, the only minimum covering set is $S_2$.

Therefore, $c(G, k) = 1$. Covering sets with $k + 1$ elements are of the form $S_2 \{v_i\}, i = 1, 2, \ldots, k + 1$. or $S_1$.

Therefore, $c(G, k + 1) = (k + 1)C_1 + 1$

The vertex covering sets with $k + 2$ elements are of the form:

$S_2 \{v_i, v_j\}$, $S_2 - \{u_i\} \bigcap N(u_i)/deg(u_i) = 3$

$S_1 \{u_i\}$, $S_1 - \{v_i\} \bigcap N(v_i)/deg(v_i) = 2$

Therefore, $c(G, k + 2) = (k + 1)C_2 + 4(p + 1) + kC_1 + 4$

The covering sets with $k + 3$ elements are of the forms:

$S_2 \{v_i, v_j, v_k\}$; $S_2 - \{u_i\} \bigcap N(u_i)/deg(u_i) = 3$

$S_1 \{u_i\}$, $S_1 - \{v_i\} \bigcap N(v_i)/deg(v_i) = 2$

Therefore, $c(G, k + 3) = (k + 1)C_3 + 4(p + 1)((k - 2)C_1) + q + kC_2 + 4((k - 2)C_1) + 4p$

The vertex covering set with $(k + 4)$ elements are of the form:

$S_2 \{v_i, v_j, v_k, v_{\ell}\}$; $S_2 - \{u_i\} \bigcap N(u_i)/deg(u_i) = 3$

$S_1 \{u_i\}$, $S_1 - \{v_i\} \bigcap N(v_i)/deg(v_i) = 2$

Therefore, $c(G, k + 4) = (k + 1)C_4 + 4(p + 1)((k - 2)C_2) + q((k - 3)C_1) + kC_3 + 4((k - 2)C_2) + 4p((k - 3)C_1) + (q + 1)$

Similarly, $c(G, k + 5) = (k + 1)C_5 + 4(p + 1)((k - 2)C_3) + q((k - 3)C_2) + kC_4$

$+ 4((k - 2)C_3) + 4p((k - 3)C_2) + (q + 1)((k - 4)C_1)$. 
\[ c(G, 2k - 1) = (k + 1) C_k - 1 + 4 (p + 1) ((k - 2) C_{k - 3}) + q ((k - 3) C_k - 4 )) + kC_k - 2 \\
+ 4 ((k - 2) C_{k - 1}) + 4p ((k - 3) C_{k - 4}) + (q + 1) ((k - 4) C_{k - 5}). \]
\[ c(G, 2k) = 2k + 1; \quad c(G, 2k + 1) = 1 \]
\[ C(G, x) = x^k + \{ (k + 1) C_1 + 1 \} x^{k + 1} + \{ (k + 1) C_2 + 4 (p + 1) + kC_1 + 4 \} x^k + 2 \\
+ \{ [ (k + 1) C_3 + 4 (p + 1) ((k - 2) C_1)) + q + kC_2 + 4 ((k - 2)C_1) + 4p ] \} x^{k + 3} \\
+ \{ [ (k + 1) C_4 + 4 (p + 1) ((k - 2) C_2)) + q ((k - 3) C_1) + kC_3 + 4 ((k - 2) C_2) + 4p ((k - 3) C_1) \} + (q + 1) \} x^k + 4 + \{ (k + 1) C_5 + 4 (p + 1) ((k - 2) C_3) + q (k - 3) \} x^k + 3 + \ldots + \]
\[ C_2 \\
+ kC_4 + 4 (k - 2) C_3 + 4p ((k - 3) C_2) + (q + 1) ((k - 4) C_1)) \} x^k + 5 + \ldots + \]
\[ \{ [(k + 1) C_{k - 1} + 4 (p + 1) ((k - 2) C_{k - 3}) + q ((k - 3) C_{k - 4}) + kC_{k - 2} + 4 ((k - 2) C_k) - 3 \} \} x^k - 4 + \ldots + \]
\[ kx^k + x^k + 4 ] + 4(p + 1) x^k + 2 [ 1 + (k - 2) C_1 x + (k - 2) C_2 x^2 + . . . + (k - 2) C_k - 3 x^{k - 3} + x^k - 2] \\
- 4 (p + 1) x^2k + kx^2k + qx^k + 3 [1 + (k - 3) C_1 x + (k - 3) C_2 x^2 + . . . + (k - 3) C_k - 4 x^{k - 4} + x^k - 3] \\
- 4x^2k + 1 \\
+ 4x^k + 2 [ 1 + (k - 2) C_1 x + (k - 2) C_2 x^2 + . . . + (k - 2) C_k - 3 x^{k - 3} + x^k - 2] - 4x^2k \\
+ 4px^k + 3 [1 + (k - 3) C_1 x + (k - 3) C_2 x^2 + . . . + (k - 3) C_k - 4 x^{k - 4} + x^k - 3] - 4px^2k \\
+ (q + 1) x^k + 4 [1 + (k - 4) C_1 x + (k - 4) C_2 x^2 + . . . + (k - 4) C_k - 5 x^{k - 5} + x^k - 4] \\
- (q + 1) x^{2k} \\
= x^k [ 1 + x]^k + 1 + 4(p + 1) x^k + 2 [ 1 + x]^k - 2 + qx^k + 3 [ 1 + x]^k - 3 + x^k + 1 \\
[1 + x]^k \\
+ 4x^k + 3 [ 1 + x]^k - 2 + 4px^k + 3 [ 1 + x]^k - 3 + (q + 1) x^k + 4 [ 1 + x]^k - 4 - x^{2k} \\
[8p + 2q + 9 + x] \\
C(G, x) = x^k [ 1 + x]^k + 1 + 4 (p + 2) x^k + 2 [ 1 + x]^k - 2 + (4p + q) x^k + 3 [ 1 + x]^k - 3 \]
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\[ +x^k + 1 - (1 + x)^k + [q + 1]x^k + 4 \cdot [1 + x]^k - 4 \cdot x^2k \cdot [8q + 2q + 9 + x] \]

\[ +x^k + 1 - (1 + x)^k + [q + 1]x^k + 4 \cdot [1 + x]^k - 4 \cdot x^2k \cdot [8q + 2q + 9 + x] \]

\[ +x^k + 1 - (1 + x)^k + [q + 1]x^k + 4 \cdot [1 + x]^k - 4 \cdot x^2k \cdot [8q + 2q + 9 + x] \]

\[ +x^k + 1 - (1 + x)^k + [q + 1]x^k + 4 \cdot [1 + x]^k - 4 \cdot x^2k \cdot [8q + 2q + 9 + x] \]

\[ +x^k + 1 - (1 + x)^k + [q + 1]x^k + 4 \cdot [1 + x]^k - 4 \cdot x^2k \cdot [8q + 2q + 9 + x] \]

Proposition 2.13
The vertex cover polynomial of the peterson graph \( P \) is
\[ C(P, x) = 5x^6 + 20x^7 + 30x^8 + 10x^9 + 10 \]

Proof
Let \( v_i \) be any be vertex of \( P \).
Then \( N(v_i) \) contains three vertices.

Let \( N(v_i) = \{v_j, v_k, v_l\} \). Choose one element say \( v_i \) \( N(v_i) \) and also the neighbouring vertices of \( v_k \) and \( v_l \). Then these constitute a set of six elements which is covering set of \( G \).

Thus, the minimum vertex covering set of \( P \) contains exactly six elements. The family of covering sets of cardinality six is of the form.

\[ \{\{v_i\} \cup \{v_j\} \cup N(v_k) \cup N(v_l) / v_j, v_k \text{ and } v_l \in N(v_i)\}. \]

There are five such covering sets of cardinality 6. Therefore, \( c(P, 6) = 5 \).
In addition to the six elements in the minimum vertex covering sets, one each of the remaining four vertices is a cover for \( P \) of cardinality seven.
Therefore, the number of covering sets with cardinality seven is \( 5(4C_1) \)
That is, \( c(P, 7) = 5(4C_1) \)

Similarly, \( c(G, 8) = 5(4C_2) \); \( c(G, 9) = 5 \left( \frac{4C_3}{2} \right) \)

Therefore, \( C(G, x) = 5x^6 + 5(4C_1)x^7 + 5(4C_2)x^8 + 5 \left( \frac{4C_3}{2} \right) x^9 + x^{10}. \)

\[ = 5x^6 + 20x^7 + 30x^8 + 10x^9 + x^{10}. \]

\[ = 5x^6 + 20x^7 + 30x^8 + 10x^9 + x^{10}. \]
Proposition 2.14
Let \( G \) be a cubic graph with 10 vertices. Then, its vertex cover polynomial is
\[
C(G, x) = 2x^5 [1 + x]^5 - x^{10}.
\]
Let \( S_1 = \{v_1, v_3, v_5, v_7, v_9\} \)
\( S_2 = \{v_2, v_4, v_6, v_8, v_{10}\} \)

Then, \( S_1 \) and \( S_2 \) are minimum vertex covering sets.
Also, \( V - S_1 \) and \( V - S_2 \) are disjoint sets.

Therefore,
\[
C(G, x) = 2[x^5 + 5C_1 x^6 + 5C_2 x^7 + 5C_3 x^8 + 5C_4 x^9] + x^{10}.
\]
\[
= 2x^5 + 10x^6 + 20x^7 + 20x^8 + 10x^9 + x^{10}
\]
\[
= 2x^5 [1 + 5C_1 x + 5C_2 x^2 + 5C_3 x^3 + 5C_4 x^4 + x^5] - x^{10}.
\]
\[
= 2x^5 [1 + x]^5 - x^{10}.
\]

Theorem 2.15
Let \( K_n \) be any complete graph with \( n \) vertices and \( G = K_n \circ K_1 \) (one corona).

Then the vertex cover polynomial of \( G \) is
\[
C(G, x) = (n + 1)x^n + n (nC_1)x^{n+1} + (n - 1)(nC_2)x^{n+2} + \ldots + 2(nC_{n-1})x^{2n-1} + x^{2n}.
\]

Figure 13
Proof

If $n = 1$, the result is trivial.

Let $n = 2$.

Then the covering sets of cardinality two are

$C(G, 2) = \{\{v_1N, v_2\}, \{v_1, v_2N\}, \{v_1, v_2\}\}.$

Similarly, $C(G, 3) = \{\{v_1N, v_1, v_2\}, \{v_1N, v_1, v_2N\}, \{v_1, v_2N\}\}$

$C(G, 4) = \{v_1N, v_1, v_2, v_2N\}$

Therefore, $C(G, 2) = 3$ ; $C(G, 3) = 4$ ; $C(G, 4) = 1$

$C(G, x) = 3x^2 + 4x^3 + x^4$ \hspace{1cm} (1)

$= (2 + 1)x^2 + 2(2C_1)x^2 + 1 + (2 - 1)(2C_2)x^2 + 2$

Therefore, the result is true for $n = 2$.

When $n = 3$, the graph and its corresponding polynomial are given below:

$C(G, 3) = \{\{v_1, v_2, v_3N\}, \{v_2, v_3, v_1N\}, \{v_1, v_3, v_2N\}, \{v_1, v_2, v_3\}\}$

$C(G, 4) = \{v_1, v_2, v_3, v_3N\} \cdot \{v_1, v_2, v_1N, v_2N\} \cdot \{v_1, v_2, v_2Nv_3N\}$

$\{v_1, v_2, v_3, v_2N\} \{v_1, v_3, v_1N, v_2N\}, \{v_1, v_2, v_2N, v_3N\}$

$\{v_1, v_2, v_3, v_1N\} \{v_2, v_3, v_1N, v_2N\}, \{v_2, v_3, v_1N, v_3N\}$

(Figure 14)

Therefore, $C(G, 4) = 9$

$C(G, 5) = \{\{v_1, v_2, v_3, v_2N, v_3N\}, \{v_1, v_2, v_3, v_1N, v_3N\}, \{v_1, v_2, v_3, v_2N, v_3N\}$

$\{v_2, v_3, v_1N, v_2N, v_3N\}; \{v_1, v_3, v_1N, v_2N, v_3N\}. \{v_1, v_2, v_1N, v_2N, v_3N\}$

Therefore, $c(G, 4) = 6$ and

$C(G, 6) = \{v_1, v_2, v_3, v_1N, v_2N, v_3N\}$ Therefore, $c(G, 6) = 1$. 
Therefore, the vertex cover polynomial is
\[ C(G, x) = 4x^3 + 9x^4 + 6x^5 + x^6 \]
\[ = (3 + 1)x^3 + 3(3C1)x^4 + (3 - 1)(3C2)x^5 + x^6 \]

Therefore, the result is true for \( n = 3 \)
We assume that the result is true for all values less than or equal to 'n'. We prove for \( n + 1 \).
Since the result is true for 'n', we have
\[ C(G, x) = (n + 1)x^n + n(nC1)x^{n+1} + (n - 1)(nC2)x^{n+2} + \ldots, 2(nC_n - 1)x^{2n} \]
\[ + x^{2n} \] —— (A)

Now, we are adding the new vertices \( v_{n+1} \) and \( v_{n+1}' \) with the existing vertices of \( K_n \circ K_1 \) in such a way that the vertex \( v_{n+1} \) is adjacent with all \( v_i, i = 1 \ldots n \) and also adjacent with \( v_{n+1}' \). Now the graph \( G = K_n + 1 \circ K \). In \( K_n \circ K_1 \); \( \beta(G) = n \) and its corresponding \( n + 1 \) covering sets are
\[ \{v_1, v_2, \ldots v_i - 1, v_i + 1, \ldots v_n, v_{n+1}'\}_{i = 1, 2, \ldots n} \text{ and } \{v_1, v_2, \ldots v_n\} \]
The \( n^2 \) vertex covering sets for \( K_n \circ K_n \) with cardinality \( (n + 1) \) are
\[ \{v_1, v_2, \ldots v_i - 1, v_i + 1, \ldots v_n, v_iN, v_jN\} \text{ and } \{v_1, v_2, \ldots v_i, \ldots v_n, v_iN, v_jN\}_{i, j = 1 \ldots n ; i \neq j} \]

For \( K_n + 1 \circ K_1 \), [In \( K_n \circ K_n \) the edges \( v_{n+1} v_i \); \( i = 1 \ldots n \) and \( v_{n+1} v_{n+1}' \) are added]
The minimum vertex covering sets with cardinality \( n + 1 \) are
\[ \{v_1, v_2, \ldots v_i - 1, v_i + 1, \ldots v_n, v_iN, v_{n+1}'\} ; i = 1 \ldots n; \]

(Figure 15)
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The vertex covering sets with cardinality $n + 2$ for $K_{n+1} \circ K_1$ are the sets given in (1) by inserting $v_{n+1}$ with each of the sets.

That is, $\{v_1, v_2, \ldots, v_i, v_{i+1}, \ldots, v_n, v_i^N, v_{n+1}\}$; $i, j = 1, \ldots, n$; $i \neq j$

Also, $\{v_1, v_2, \ldots, v_{n+1}, v_i^N\}$; $i = 1, \ldots, n$ and $\{v_1, v_2, \ldots, v_n, v_{n+1}, v_j^N\}$; $j = 1, \ldots, n+1$

Thus, the $n^2$ covering sets of $K_n \circ K_1$ [including the vertex $v_{n+1}$] and the additional $2n+1$ vertex covering sets from the family of covering sets of $K_{n+1} \circ K_1$ with cardinality $n+2$ form the covering sets of cardinality, $n + 2$, for $K_{n+1} \circ K_1$.

Therefore, $c(G, n + 2) = n(nC_1) + 2n + 1$.

Proceeding this way, $2n + 1$ elements can be taken from $2(n + 1)$ elements in $2(n + 1) C_1$ ways. Therefore, $c(G, 2n+1) = 2(n+1)$. Also $c(G, 2n + 2) = 1$.

Therefore, the vertex cover polynomial is

$$C_1(G, x) = (n + 2) x^{n+1} + (n(nC_1) + 2n + 1) x^{n+2} + 2 + 2 (n + 1) x^{2n+1} + x^{2n} + 2$$

$$= (n + 1 + 1) x^{n+1} + (n + 1 + 2) x^{n+2} + \ldots + 2 (n + 1) x^{2n+1} + x^{2n+2}$$

$$= (n + 1 + 1) x^{n+1} + (n + 1) (n + 1) C_1 x^{n+2} + \ldots + 2 (n + 1) x^{2n+1} + x^{2n+2}$$

$$= (n+1) C_1 x^{n+2} + \ldots + 2 (n+1) C_1 x^{2n+1} + x^{2n+2} + (n+1) C_1 x^{2(n+1)}$$

Hence the theorem.

References