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On Vertex-Cover Polynomial on Some Standard Graphs

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Abstract

The vertex cover Polynomial of a graph G of order n has been already introduced in [3]. It is defined as the polynomial, $C(G, i) = \sum_{i=\beta(G)}^{|v(G)|} c(G, i)x^i$ where $c(G, i)$ is the number of vertex covering sets of G of size i and $\beta(G)$ is the covering number of G. We obtain some properties of $C(G, x)$ and its coefficients for some standard graphs. Also, we compute the vertex cover polynomials for $K_n - \{v\}$, the product graph $K_m \times K_n$, the net graph, the Peterson graph, the cubic graph and $K_n \circ K_1$.

Keywords: Vetex covering set, vertex covering number, vertex cover polynomial.

Introduction

Let $G = (V, E)$ be a simple graph. For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V / uv \in E\}$ and the closed neighborhood of v is the set $N[v] = N(v)\ \chi \{v\}$. For a set $S\ \phi V$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S\ \phi V$ is a vertex covering of G if every edge $uv \in E$ is adjacent to at least one vertex in S. The vertex covering number $\beta(G)$ is the minimum cardinality of the vertex covering sets in G. A vertex covering set with cardinality $\beta(G)$ is called a β- set. Let $C(G, i)$ be the family of vertex covering sets of G with cardinality i and let $c(G, i) = |C(G, i)|$. The polynomial, $C(G, x) = \sum_{i=\beta(G)}^{|v(G)|} c(G, i)\, x^{i}$ is defined as the vertex cover polynomial of G. In [3], many properties of the vertex cover polynomials have been studied and derived the vertex cover polynomials for some standard graphs. In this paper, we find the expression for vertex cover polynomial of disjoint union of graphs. Also, we find vertex cover polynomials for some standard graphs such as the Peterson graph, Net graph etc.

The number of edges incident to the vertex v of a graph G is called the degree of the vertex v in G. It is denoted by deg (v). $\delta(G)$ and $\Delta(G)$ are the minimum and maximum of the degree of all the vertices in G respectively. The composition $G_1[G_2]$ as having $V = V_1 \times V_2$ and $u = (u_1, v_1)$ and $v = (u_2, v_2)$ are adjacent if u_1 is adjacent to u_2 in G_1 or $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2. The product $G_1 \times G_2$ as having $V = V_1 \times V_2$ and $u = (u_1, v_1)$ and $v = (u_2, v_2)$ are adjacent if $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2 or $v_1 = v_2$ and u_1 is adjacent to u_2 in G_1.

Vertex Cover Polynomials

Definition 2.1: Let C(G, i) be the family of all vertex covering sets of G with cardinality i and let $c(G, i) = |C(G, i)|$. The vertex cover polynomial of G is defiend as

$$C(G, x) = \sum_{i=\beta(G)}^{|v(G)|} c(G, i)\, x^{i}.$$

In [3] the vertex cover polynomial of K_n is obtained as $C(K_n, x) = nx^{n-1} + x^n$.

Theorem: 2.2 Let K_n be the complete graph with n vertices.

Then, $C(K_n - \{v\}, x) = \dfrac{1}{n}\dfrac{d}{dx}[C(K_n, x)]$.

Proof: As K_n is complete with n vertices, $K_n - \{v\}$ is complete with $n-1$ vertices. We have,

$$C(K_n, x) = x^n\left(1 + \frac{n}{x}\right) \qquad \text{(i)}$$

and

$$C(K_n - \{v\}, x) = x^{n-1}\left(1 + \frac{n-1}{x}\right) \qquad \text{(ii)}$$

Differentiating (i) we get,

$$\frac{d}{dx}\,[C\,(K_n, x)] \;=\; \frac{d}{dx}\left[x^n\left(1+\frac{n}{x}\right)\right]$$

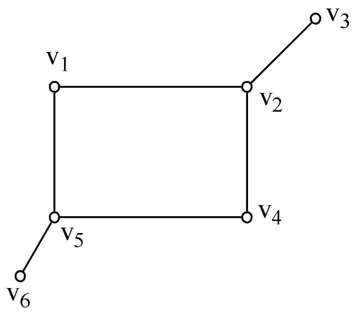
$$=\; n\,x^{n-1}\left(1+\frac{n-1}{x}\right)$$

Therefore, $\frac{1}{n}\,\frac{d}{dx}\;[C\,(K_n, x)] \;=\; x^{n-1}\left(1+\frac{n-1}{x}\right)$

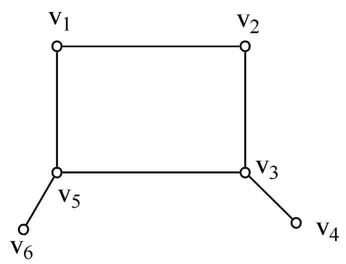
$= C\,(K_n - \{v\}, x)$ [by (ii)]

Therefore, $C\,(K_n - \{v\}, x) = \frac{1}{n}\,\frac{d}{dx}\;[C\,(K_n, x)]$ □

Remark: 2.3
If G_1 and G_2 are any two graphs with the same degree sequence, then $C(G_1, x)$ and $C(G_2, x)$ need not be the same.
(eg)



(Figure 1 a) G_1 (3, 3, 2, 2, 1, 1)

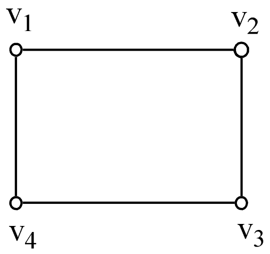


(Figure 1 b) G_2 (3, 3, 2, 2, 1, 1)

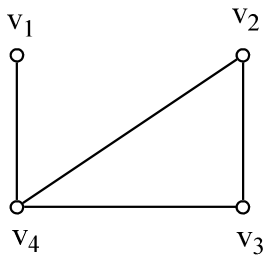
$C\,(G_1, x) = x^2 + 4x^3 + 9x^4 + 6x^5 + x^6$; $C\,(G_2, x) = 4x^3 + 9x^4 + 6x^5 + x^6$
therefore, $C\,(G_1, x) \neq C\,(G_2, x)$

Definition: 2.4. Let G_1 and G_2 be any two graphs. If $C(G_1, x) = C\,(G_2, x)$, then G_1 and G_2 are said to be C - equivalence graphs.

Note : 2.5
Two non isomorphic graphs can be C - equivalance. (eg)



(Figure 2 a)



(Figure 2 b)

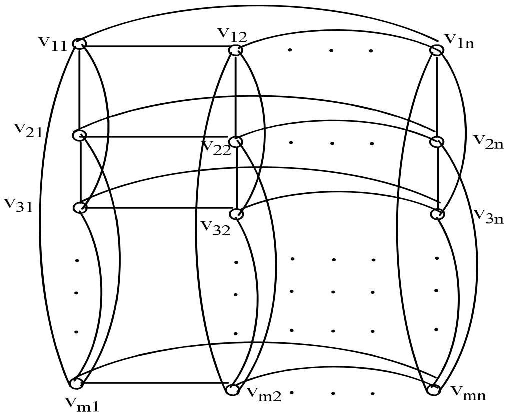
$C(G_1, x) = 2x^2 + 4x^3 + x^4$, $\quad$ $C(G_2, x) = 2x^2 + 4x^3 + x^4$ □

Theorem: 2.6
Let K_m and K_n be the complete graphs with m, n vertices respectively. Then the vertex cover polynomial of the cross product $K_m \times K_n$ is

$$C(K_m \times K_n, x) = nC_0\,[mC_{m-1}.\,(m-1)\,C_{m-2}\ldots(n\text{ terms})]\,x^{n(m-1)}$$
$$+\ nC_1\,[mC_{m-1}.\,(m-1)\,C_{m-2}\ldots(n-1)\text{ terms})]\,x^{n(m-1)+1}$$
$$+\ nC_2\,[mC_{m-1}\,.\,(m-1)\,C_{m-2}\ldots.(n-2)\text{ terms}]\,x^{n(m-1)+2}$$
$$+\ \ldots..+\ nC_{n-1}\,(mC_{m-1})\,x^{nm-1} + nC_n\,x^{nm}$$

Proof: $K_m \times K_n$ is the graph with vertex set $V = \{v_{ij} = (u_i, v_j) / u_i \in K_m$ and $v_j \in K_n$, $i = 1.....m,\ j = 1....n\}$ and $|V| = mn$. Now the vertices of $G = K_m \times K_n$ can be arranged in the form of a matrix of order m x n.

Now, each column in the graph given in figure 3 represents the graph K_m. We need any m - 1 elements to cover all edges of Column-1. In a similar way, we have to select m – 1 elements in each column to cover all edges of G. Now the vertices of G in the first column are denoted by $v_{11}, v_{21} \cdot v_{m1}$. The minimum vertex covering set of G is any (m - 1) elements in the first column among the vertices, $v_{i1}, i = 1...m$. It can be selected in m ways. Suppose the element v_{k1} is not selected in the first column, then the element v_{k2} should be selected in the second column in the corresponding row, and (m – 2) elements have to be selected from the remaining (m – 1) elements in the second column other than v_{k2}. Similary, suppose the element v_{t2} was not selected in the second column, then the element v_{t3} should be selected in the third column.



K_m x K_n

Figure 3

Thus two elements are fixed in the third column. The m – 3 elements in column 3 have to be selected from m – 2 elements and so on.

Therefore, c (G, n (m – 1)) = nC_0 [mC_{m-1} + (m – 1) C_{m-2} +.... (n terms)]

Similarly, c (G, n (m – 1) +1) = nC_1 [mC_{m-1} + (m-1) C_{m-2} + ... (n – 1) terms] c ((G, n (m – 1) + 2) = nC_2 [mC_{m-1} + (m - 1) C_{m-2} +.... (n-2) terms] , . . . , c (G, mn – 1) = nC_{n-1} [mC_{m-1}] & c (G, mn) = 1

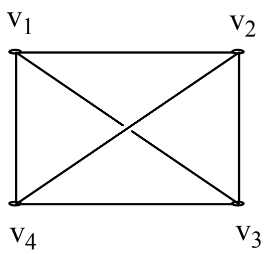
Therefore,. C (K_m x K_n, x) = nC_0 [mC_{m-1} . (m – 1) C_{m-2} . (n terms)] $x^{n(m-1)}$

+ nC_1 [mC_{m-1} . (m – 1)C_{m-2} . ... (n – 1) terms)] $x^{n(m-1)+1}$

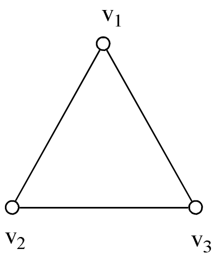
+ nC_2 [mC_{m-1} . (m – 1) C_{m-2} ... (n – 2) terms)] $x^{n(m-1)+2}$

+ . . . . + nC_{n-1} [mC_{m-1}]x^{nm-1} + nC_n x^{nm}. □

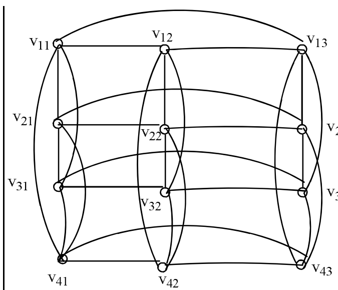
Example: 2.7



$G_1 = K_4$
(Figure 4 a)



$G_2 = K_3$
(Figure 4 b)



K_4 x K_3
(Figure 4 c)

Here m = 4 and n = 3

$C(G, x) = 3C_0 [4C_3 . 3C_2 . 2C_1]\ x^9 + 3C_1 [4C_3 . 3C_2]\ x^{10} + 3C_2 [4C_3]\ x^{11} + 3C_3\ x^{12}$

$C(G, x) = 24x^9 + 36x^{10} + 12x^{11} + x^{12}$

Corollary 2.8 : For the complete graphs, K_m and K_n

$$C(K_m [K_n], x) = x^{mn}\left[1 + \frac{mn}{x}\right]$$

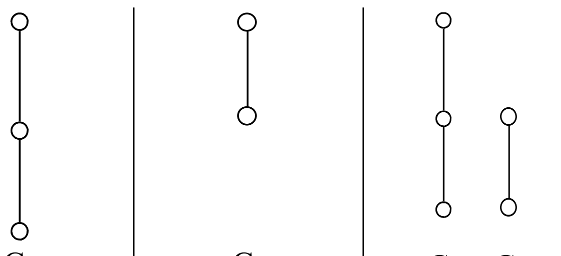
Proof: Composition of K_m and K_n is a complete graph with m n vertices. Then $C(G, x) = mn\ x^{mn-1} + x^{mn} = x^{mn}\left[1 + \frac{mn}{x}\right]$, where $G = K_m \circ [K_n]$, since for a complete graph of order n, the polynomial is $nx^{n-1} + x^n$.

Note: 2. 9
If G_1 and G_2 are complete, then $C(G_1 [G_2], x) = C(G_2 [G_1], x)$

Theorem 2.10: If G_1 and G_2 are any two disjoint graphs, then
$C(G_1 \vee G_2, x) = C(G_1, x).\ C(G_2, x)$.

Proof
Since G_1 and G_2 are any two graphs with $G_1 \cap G_2 = \varphi$. The selection of vertex covering set of G_1 does not affect the terms of G_2.

Therefore, $C(G_1 \vee G_2, x) = C(G_1\ x) . C(G_2, x)$. (eg)



G_1 (Figure 5 a) | G_2 (Figure 5 b) | $G_1\ \omega\ G_1$ (Figure 5 c)

$C(G_1, x) = x + 3x^2 + x^3$; $C(G_2, x) = 2x + x^2$

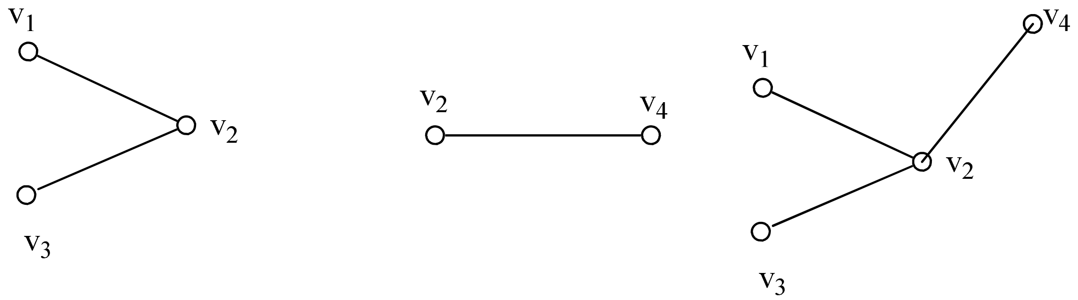
$C(G_1 \vee G_2, x) = x^2 + 7x^3 + 11x^4 + 6x^5 + x^6 = C(G_1, x) . C(G_2, x)$ □

Note

If $G_1 \cap G_2 \neq \varphi$ then

$C(G_1 \vee G_2, x) \neq C(G_1, x) \,.\, C(G_2, x)$

Example 2.11



(G_1) (Figure 6a) | (G_2) (Figure 6 b) | ($G_1 \omega G_2$) (Figure 6 c)

$C(G_2, x) = x + 3x^2 + x^3$; $C(G_2, x) = 2x^2 + x^2$

$C(G_1 \omega G_2, x) = x + 3x^2 + 4x^3 + x^4$ (i)

$C(G_1, x) \cdot C(G_2, x) = 2x^2 + x^3 + 5x^4 + x^5$ (ii)

From (i) & (ii) $C(G_1 \vee G_2, x) \neq C(G_1, x) \,.\, C(G_2, x)$

Theorem : 2.12

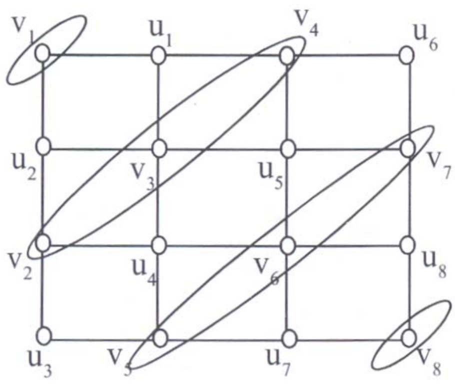
The vertex cover polynomial of any net graph G with n x n vertices ($n > 2$) is

i) $C(G, x) = 2x^k [1 + x]^k + 4x^{k+1} [1 + x]^{k-2} + 4(n-2) x^{k+3} [1 + x]^{k-3} + (n-2)^2 x^{k+3} [1 + x]^{k-4} - x^{2k-1} [n^2 + 2n + x]$

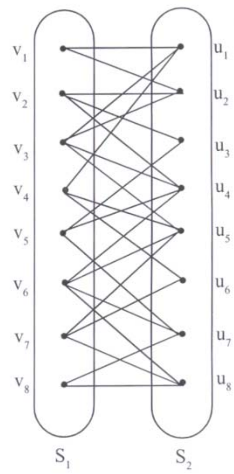
if n is even, where $k = \left\lfloor \frac{n^2}{2} \right\rfloor$

ii) $C(G, x) = x^k [1 + x]^{k+1} + 4(p + 2) x^{k+2} [1 + x]^{k-2} + (4p + q) x^{k+3} [1+x]^{k-3} + x^{k+1} [1 + x]^k + (q + 1) x^{k+4} [1 + x]^{k-4} - x^{2k} (8p + 2q + 9 + x)$. for any odd n, where $k = \left\lfloor \frac{n^2}{2} \right\rfloor$; $T = n - 2$; $p = \left\lfloor \frac{T}{2} \right\rfloor$; $q = \left\lfloor \frac{T^2}{2} \right\rfloor$ Proof:

(i) The net graph with n x n (n is even) can be converted into bipartite graph as follows. A graph of 4 x 4 vertices and its corresponding bipartite is shown below in (figure 7 & 8).



(Figure 7)



(Figure 8)

Let $S_1 = \{v_1, v_2, ... v_k\}$ and

$S_2 = \{u_1, u_2, ... u_k\}$

where $k = \left\lfloor \frac{n^2}{2} \right\rfloor$.

Therefore, $|S_1| = |S_2| = k$.

In S_1 and S_2, two vertices are of degree 2; 2(n-2) vertices are of degree 3 and $\frac{(n-2)^2}{2}$ vertices of degree 4. The minimum vertex covering set with cardinality k is S_1 or S_2. Therefore, $c(G, k) = 2$. The vertex covering set with cardinality $k+1$ are any one of the following form :

$S_1 \chi \{u_i\}$, $S_2 \chi \{v_i\}$;

$S_1 - \{v_i\} \chi N(v_i)$; $S_2 - \{u_i\} \chi N(u_i) / d(u_i) = d(v_i) = 2$

Therefore, $c(G, k+1) = 2(kC_1) + 4$

The sets having cardinality $k+2$ are any one of the following form :

$S_1 \chi \{u_i, u_j\}$, $S_2 \chi \{v_i, v_j\} / u_i, \tilde{u_j}\ S_2 \ \&\ v_i, \tilde{v_j}\ S_1,$

$S_1 - \{v_i\} \chi N(v_i) \chi \{u_i\} / u_i \notin N(v_i) \ \&\ deg(v_i) = 2,$

$S_2 - \{u_i\} \chi N(u_i) \chi \{v_i\} / v_i \notin N(u_i) \ \&\ deg(u_i) = 2,$

$S_1 - \{v_i\} \chi N(v_i) / deg(v_i) = 3,$

$S_2 - \{u_i\} \chi N(u_i) / deg(u_i) = 3,$

Therefore, $c(G, k+2) = 2(kC_2) + 4((k-2)C_1) + 4(n-2)$

The covering sets with $k+3$ elements are any one of the following form

$S_1 \chi \ \{u_i, u_j, u_k\}$, $S_2 \chi \ \{v_i, v_j, v_k\}$,
$S_1 - \{v_i\} \chi \ N(v_i) \chi \{u_i, u_j\} / \{u_i, u_j\} \notin N(v_i)$ & $\deg(v_i) = 2$,
$S_2 - \{u_i\} \chi N(u_i) \chi \{v_i, v_j\} / \{v_i, v_j\} \notin N(u_i)$ & $\deg(u_i) = 2$,
$S_1 - \{v_i\} \chi N(v_i) \chi \{u_i\} / u_i \notin N(v_i)$ & $\deg(v_i) = 3$,
$S_2 - \{u_i\} \chi N(u_i) \chi \{v_i\} / v_i \notin N(u_i)$ & $\deg(u_i) = 3$,
$S_1 - \{v_i\} \chi N(v_i) / \deg(v_i) = 4$,
$S_2 - \{u_i\} \chi N(u_i) / \deg\{u_i\} = 4$

Therefore , $c(G, k+3) = 2(kC_3) + 4((k-2)C_2) + 4(n-2)((k-3)C_1) + (n-2)^2$

In a similar way,

$c(G, k+4) = 2(kC_4) + 4((k-2)C_3) + 4(n-2) + ((k-3)C_2) + (n-2)^2((k-4)C_1)$

$c(G, k+5) = 2(kC_5) + 4(k-2)C_4) + 4(n-2)((k-3)C_3) + (n-2)^2((k-4)C_2$;

$c(G, 2k-3) = 2(kC_{K-3}) + 4((k-2)C_{k-4}) + 4(n-2)(k-3)C_{k-5} + (n-2)^2(k-4)C_{k-6})$

$c(G, 2k-2) = 2(kC_{k-2}) + 4((k-2)C_{k-3}) + 4(n-2)((k-3)C_{k-4}) + (n-2)^2((k-4)C_{k-5})$

$c(G, 2k-1) = 2(kC_{K-1})$ and $c(G, 2k) = 1$

Therefore, $C(G, x) = 2x^k + 2[kC_1 x^{k+1} + kC_2 x^{k+2} + kC_3 x^{k+3} + \ldots + kC_{k-2} x^{2k-2} + kC_{k-1} x^{2k-1} + kC_k x^{2k}] - x^{2k} + 4[x^{k+1} + (k-2)C_1 x^{k+2} + (k-2)C_2 x^{k+3} + \ldots +$
$(k-2)C_{k-4} x^{2k-3} + (k-2)C_{k-3} x^{2k-2}] + 4(n-2)[x^{k+2} + (k-3)C_1 x^{k+3} + (k-3)C_2 x^{k+4}$
$+ \ldots + (k-3)C_{k-5} x^{2k-3} + (k-3)C_{k-4} x^{2k-2}] + (n-2)^2[x^{k+3} + (k-4)C_1 x^{k+4}$
$+ (k-4)C_2 x^{k+5} + \ldots + (k-4)C_{k-6} x^{2k-3} + (k-4)C_{k-5} x^{2k-2}]$
$= 2x^k[1 + kC_1 x + kC_2 x^2 + \ldots + kC_{k-1} x^{k-1} + kC_k x^k] - x^{2k}$
$+ 4x^{k+1}[1 + (k-2)C_1 x + (k-2)C_2 x^2 + \ldots + (k-2)C_{k-4} x^{k-4} + (k-2)C_{k-3} x^{k-3}]$
$+ 4(n-2)x^{k+2}[1 + (k-3)C_1 x + (k-3)C_2 x^2 + \ldots + (k-3)C_{k-5} x^{k-5} + (k-5)C_{k-4} x^{k-4}] + (n-2)^2 x^{k+3}[1 + (k-4)C_1 x + (k-4)C_2 x^2 + \ldots + (k-4)C_{k-6} x^{k-6} + (k-4)C_{k-5} x^{k-5}]$

= $2x^{k} [1 + x]^{k} - x^{2k} + 4x^{k+1} [1 + x]^{k-2} - 4x^{2k-1} + 4(n - 2) x^{k+2} [1 + x]^{k-3} - 4 (n - 2) x^{2k-1}$

+ $(n - 2)^{2} x^{k+2} [1 + x]^{k-4} - (n - 2)^{2} x^{2k-1}$

= $2x^{k} [1 + x]^{k} + 4x^{k+1} [1 + x]^{k+2} + 4 (n - 2) x^{k+2} [1 + x]^{k-3} + (n - 2)^{2} x^{k+3} [1 + x]^{k-4} - x^{2k-1} [n^{2} + 2n + x]$.

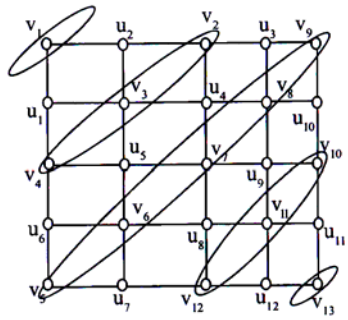


Figure 9

(ii) The net graph with n x n vertices (n is odd) can be converted as a bipartite graph as follows

For example, a graph of 5 x 5 vertices and its corresponding bigraph is shown below in figure 9 & 10

Let $S_1 = \{(v_1, v_2, v_3 .... v_{k+1})\}$

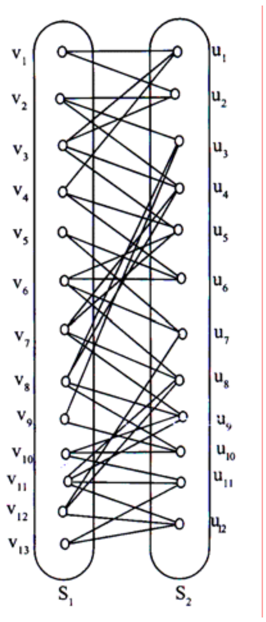


Figure 10

$S_2 = \{(u_1, u_2 \ldots\ldots u_k)$ where $k = \left\lfloor \frac{n^2}{2} \right\rfloor$

Let $T = n-2$; $p = \left\lfloor \frac{T}{2} \right\rfloor$ & $q = \left\lfloor \frac{T^2}{2} \right\rfloor$

In S_1; 4 vertices are of degree 2, 4p vertices are of degree 3 and q +1 vertices are of degree 4.

In S_2, no vertices are of degree 2. 4 (p +1) vertices are of degree 3 and q vertices are of degree 4. Also, $|S_1| = k + 1$; $|S_2| = k$.

Clearly, the only minimum covering set is S_2;

Therefore, c(G, k) = 1. Covering sets with k + 1 elements are of the form

$S_2 \chi \{v_i\}$, $_{i = 1, 2, \ldots\, k+1.}$ or S_1.

Therefore, $c(G, k+1) = (k+1)\, C_1 + 1$

The vertex covering sets with k +2 elements are of the form :

$S_2 \chi \{v_i, v_j\}$, $S_2 - \{u_i\} \chi N(u_i)/ \deg(u_i) = 3$

$S_1 \chi \{u_i\}$, $S_1 - \{v_i\} \chi N(v_i) / \deg(v_i) = 2$

Therefore, $c(G, k+2) = (k+1)\, C_2 + 4(p+1) + kC_1 + 4$

The covering sets with k + 3 elements are of the forms :

$S_2 \chi \{v_i, v_j, v_k\}$;$S_2 - \{u_i\} \chi N(u_i) \chi \{v_i\}/ v_i \notin N(u_i)$ & $\deg(u_i) = 3$

$S_2 - \{u_i\} \chi N(u_i)(/ \deg(u_i) = 4$; $S_1 \chi \{u_i, u_j\}$

$S_1 - \{v_i\} \chi N(v_i) \chi \{u_i\} / u_i \notin N(v_i)$ & $\deg v_i) = 2$

$S_1 - \{v_i\} \chi N(v_i)/ \deg(v_i) = 3$

Therefore, $c(G, k+3) = (k+1)\, C_3 + 4(p+1)((k-2)\, C_1) + q + kC_2 + 4((k-2)\, C_1) + 4p$

The vertex covering set with (k + 4) elements are of the form :

$S_2 \chi \{v_i, v_j, v_k, v_s\}$; $S_2 - \{u_i\} \chi N(u_i) \chi \{v_i, v_j\}/ \{v_i, v_j\} \notin N(u_i) \deg(u_i) = 3$,

$S_2 - \{u_i\} \chi N(u_i) \chi \{v_i\}/v_i \notin N(u_i)$ & $\deg(u_i) = 4$; $S_1 \chi \{u_i, u_j, u_k\}$,

$S_1 - \{v_i\} \chi N(v_i) \chi \{u_i, u_j\}/ (u_i, u_j) \notin N(v_i)$ & $\deg(v_i) = 2$,

$S_1 - \{v_i\} \chi N(v_i) \chi \{u_i\}/u_i \notin N(v_i)$ & $\deg(v_i) = 3$,

$S_1 - \{v_i\} \chi N(v_i)/ \deg\{v_i\} = 4$

Therefore, $c(G, k+4) = (k+1)\, C_4 + 4(p+1)\ ((k-2)\, C_2) + q((k-3)\, C_1) + kC_3 + 4((k-2)\, C_2) + 4p((k-3)\, C_1) + (q+1)$

Similarly, $c(G, k+5) = (k+1)\, C_5 + 4(p+1)((k-2)\, C_3) + q\ ((k-3)\, C_2) + kC_4 + 4((k-2)\, C_3) + 4p((k-3)\, C_2) + (q+1)((k-4)\, C_1)$.

$c(G, 2k-1) = (k+1) C_{k-1} + 4(p+1)((k-2) C_{k-3}) + q((k-3) C_{k-4})) + kC_{k-2}$
$+ 4((k-2) C_{k-1}) + 4p((k-3) C_{k-4}) + (q+1)((k-4) C_{k-5})$.

$c(G, 2k) = 2k+1; \quad c(G, 2k+1) = 1$

$\therefore C(G, x) = x^k + \{(k+1) C_1 + 1\} x^{k+1} + [(k+1) C_2 + 4(p+1) + kC_1 + 4] x^{k+2}$
$+ [(k+1) C_3 + 4(p+1)((k-2)C_1) + q + kC_2 + 4((k-2)C_1) + 4p] x^{k+3}$
$+ [(k+1) C_4 + 4(p+1)((k-2) C_2) + q((k-3) C_1) + kC_3 + 4((k-2) C_2)$
$+ 4p((k-3)C_1) + (q+1)] x^{k+4} + [(k+1) C_5 + 4(p+1)((k-2) C_3) + q(k-3) C_2$
$+ kC_4 + 4(k-2) C_3 + 4p((k-3) C_2) + (q+1)((k-4) C_1)] x^{k+5} + \ldots +$
$[(k+1) C_{k-1} + 4(p+1)((k-2) C_{k-3}) + q((k-3) C_{k-4}) + kC_{k-2} + 4((k-2) C_{k-3})$
$+ 4p((k-3) C_{k-4}) + (q+1)((k-4) C_{k-5}) x^{2k-1} + [(k+1) C_k + kC_{k-1}] x^{2k} + x^{2k+4}$

$= x^k [1 + (k+1) C_1 x + (k+1) C_2 x^2 + \ldots + (k+1) C_k x^k + x^{k4}]$
$+ 4(p+1) x^{k+2} [1 + (k-2) C_1 x + (k-2) C_2 x^2 + . + (k-2) C_{k-3} x^{k-3} + x^{k-2}]$
$- 4(p+1) x^{2k} + kx^{2k} + qx^{k+3}[1 + (k-3) C_1 x + (k-3) C_2 x^2 + \ldots + (k-3) C_{k-4} x^{k-4} + x^{k-3}]$
$- qx^{2k} + x^{k+1} [1 + kC_1 x + kC_2 x^2 + + kC_{k-2} x^{k-2} + kx^{k-1} + x^k] - kx^{2k} - x^{2k+1}$
$+ 4x^{k+2} [1 + (k-2) C_1 x + (k-2) C_2 x^2 + . + (k-2) C_{k-3} x^{k-3} + x^{k-2}] - 4x^{2k}$
$+ 4px^{k+3} [1 + (k-3) C_1 x + (k-3) C_2 x^2 + . . + (k-3) C_{k-4} x^{k-4} + x^{k-3}] - 4px^{2k}$
$+ (q+1) x^{k+4} [1 + (k-4) C_1 x + (k-4) C_2 x^2 + \ldots + (k-4) C_{k-5} x^{k-5} + x^{k-4}] - (q+1) x^{2k}$

$= x^k [1+x]^{k+1} + 4(p+1) x^{k+2} [1+x]^{k-2} + qx^{k+3} [1+x]^{k-3} + x^{k+1} [1+x\}^k$
$+ 4x^{k+3} [1+x]^{k-2} + 4px^{k+3} [1+x]^{k-3} + (q+1) x^{k+4} [1+x]^{k-4} - x^{2k} [8P + 2q + 9 + x]$

$C(G, x) = x^k [1+x]^{k+1} + 4(p+2) x^{k+2} [1+x]^{k-2} + (4p+q) x^{k+3} [1+x]^{k-3}$

$+ x^{k+1}\ (1 + x)^k\ + [q + 1]\ x^{k+4}\ [1 + x]^{k-4} - x^{2k}\ [8P + 2q + 9 + x]$ □

Proposition 2.13

The vertex cover polynomial of the peterson graph P is

$C(P, x) = 5x^6 + 20x^7 + 30x^8 + 10x^9 + 10$

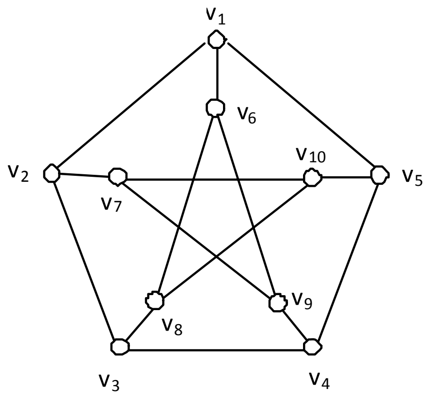
Proof

Let v_i be any be vertex of P.

Then $N(v_i)$ contains three vertices.

Let $N(v_i) = \{v_j, v_k, v_l\}$. Choose one element say $v_i^{\sim}$ $N(v_i)$ and also the neighbouring vertices of v_k and v_l. Then these constitute a set of six elements which is covering set of G.

Thus, the minimum vertex covering set of P contains exactly six elements. The family of covering sets of cordinality 6 is of the form.



(Figure 11)

$\{\{v_i\}\ \chi\{v_j\}\ \chi N(v_k)\ \chi N(v_l) / v_j, v_k$ and $v_l^{\sim} N(v_i)$.

There are five such covering sets of cardinality 6. Therefore, $c(P, 6) = 5$.

In addition to the six elements in the minimum vertex covering sets, one each of the remaining four vertices is a cover for P of cardinality seven.

Therefore, the number of covering sets with cardinality seven is $5(4C_1)$

That is, $c(P, 7) = 5(4C_1)$

Similarly, $c(G, 8) = 5(4C_2)$; $c(G, 9) = 5\left(\frac{4C_3}{2}\right)$

Therefore, $C(G, x) = 5x^6 + 5(4C_1)x^7 + 5(4C_2)x^8 + 5\left(\frac{4C_3}{2}\right)x^9 + x^{10}$.

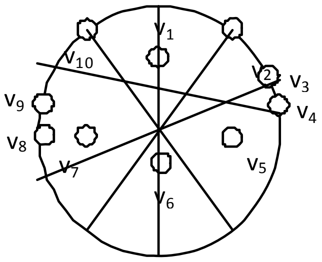
$= 5x^6 + 20x^7 + 30x^8 + 10x^9 + x^{10}$. □

Proposition 2.14
Let G be a cubic graph with 10 vertices. Then, its vertex cover polynomial is

$C(G, x) = 2x^5 [1 + x]^5 - x^{10}$.

Let $S_1 = \{v_1, v_3, v_5, v_7, v_9\}$

$S_2 = \{v_2, v_4, v_6, v_8, v_{10}\}$



(Figure 12)

Then, S_1 and S_2 are minimum vertex covering sets .

Also, $V - S_1$ and $V - S_2$ are disjoint sets.

Therefore, $C(G, x) = 2[x^5 + 5C_1 x^6 + 5C_2 x^7 + 5C_3 x^8 + 5C_4 x^9] + x^{10}$.

$= 2x^5 + 10x^6 + 20x^7 + 20x^8 + 10x^9 + x^{10}$

$= 2x^5 [1 + 5C_1 x + 5C_2 x^2 + 5C_3 x^3 + 5C_4 x^4 + x^5] - x^{10}$.

$= 2x^5 [1 + x]^5 - x^{10}$ □

Theorem 2.15
Let K_n be any complete graph with n vertices and $G = K_n \circ K_1$ (one corona). Then the vertex cover polynomial of G is $C(G, x) = (n + 1) x^n + n (nC_1) x^{n+1} + (n - 1)(nC_2) x^{n+2} + . , + 2 (nC_{n-1}) x^{2n-1} + x^{2n}$. n. 1

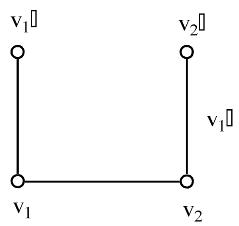


Figure 13

Proof

If $n = 1$, the result is trivial.

Let $n = 2$.

Then the covering sets of cardinality two are

$C(G, 2) = \{\{v_1N, v_2\}, \{v_1, v_2N\}, \{v_1, v_2\}\}$.

Similarly, $C(G, 3) = \{\{v_1N, v_1, v_2\}, \{v_1N, v_1, v_2N\}, \{v_1N, v_2, v_2N\}, \{v_1, v_2, v_2N\}\}$

$C(G, 4) = \{v_1N, v_1, v_2, v_2N\}$

Therefore, $C(G, 2) = 3$; $C(G, 3) = 4$; $C(G, 4) = 1$

$$C(G, x) = 3x^2 + 4x^3 + x^4 \quad (1)$$

$$= (2 + 1)\, x^2 + 2\,(2C_1)\, x^{2+1} + (2 - 1)\,(2\,C_2)\, x^{2 \text{ x } 2}$$

Therefore, the result is true for $n = 2$.

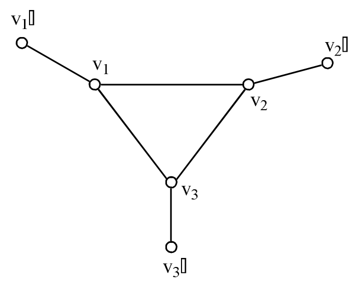
When $n = 3$, the graph and its corresponding polynomial are given below :

$C(G, 3) = \{\{v_1, v_2, v_3N\}, \{v_2, v_3, v_1N\}, \{v_1, v_3, v_2N\} ; \{v_1, v_2, v_3\}\}$

$C(G, 4) = \{v_1, v_2, v_3, v_3N\}.\{v_1, v_2, v_1N, v_2N)$. $\{v_1, v_2, v_2Nv_3N\}$

$\{v_1, v_2, v_3, v_2N\}$ $\{v_1, v_3, v_1N, v_2N\}$, $\{v_1, v_3, v_2N, v_3N\}$

$\{v_1, v_2, v_3, v_1N\}$ $\{v_2, v_3, v_1N, v_2N\}$, $\{v_2, v_3, v_1N, v_3N\}$ }



(Figure 14)

Therefore, $C(G, 4) = 9$

$C(G, 5) = \{\{v_1, v_2, v_3, v_2N, v_3N\}$. $\{v_1, v_2, v_3, v_1N, v_3N\}$. $\{v_1, v_2, v_3, v_2N, v_3N\}$

$\{v_2, v_3, v_1N, v_2N, v_3N\}$; $\{v_1, v_3, v_1N, v_2N, v_3N\}$. $\{v_1, v_2, v_1N, v_2N, v_3N\}\}$

Therefore, $c(G, 4) = 6$ and

$C(G, 6) = \{v_1, v_2, v_3, v_1N, v_2N, v_3N\}$ Therefore, $c(G, 6) = 1$.

Therefore, the vertex cover polynomial is

$C(G, x) = 4x^3 + 9x^4 + 6x^5 + x^6$

$= (3 + 1)\, x^3 + 3\,(3C_1)\, x^{3+1} + (3 - 1)\,(3C_2)\, x^{3+2} + x^{2x3}$

Therefore, the result is true for n = 3

We assume that the result is true for all values less than or equal to 'n'. We prove for n + 1.

Since the result is true for 'n', we have

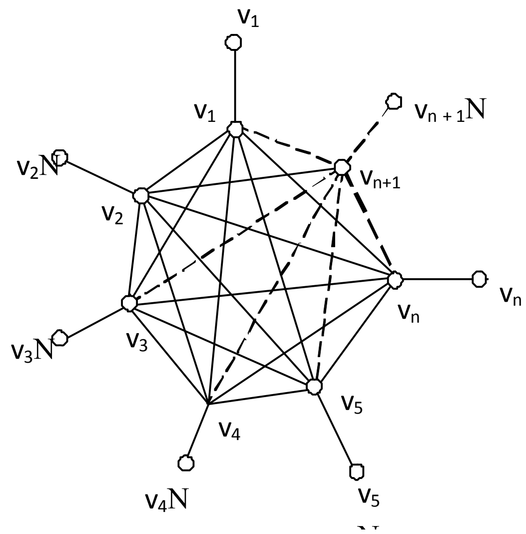
$C(G, x) = (n + 1)\, x^n + n\,(nC_1)\, x^{n+1} + (n - 1)\,(nC_2)\, x^{n+2} + \ldots, 2\,(nC_{n-1})\, x^{2n-1} + x^{2n}$ —— (A)

Now, we are adding the new vertices v_{n+1} and $v_{n+1}N$ with the existing vertices of K_n o K_1 in such a way that the vertex v_{n+1} is adjacent with all $v_{i,\ i = 1\,..\,n}$ and also adjacent with $v_{n+1}N$. Now the graph G is G = K_{n+1} o K. In K_n o K_1 ; β(G) = n and its corresponding n + 1 covering sets are

$\{v_1, v_2 \ldots v_{i-1}, v_{i+1}, \ldots v_n, v_{iN}\}_{i = 1,2,\ldots n}$ and $\{v_1, v_2, \ldots v_n\}$

The n^2 vertex covering sets for K_n o K_n with cardinality (n + 1) are

$\{v_1, v_2 \ldots v_{i-1}, v_{i+1}, \ldots v_n, v_iN, v_jN\}$ and $\{v_1, v_2 \ldots v_i, \ldots v_n, v_{iN}\} /_{\ i, j = 1 \ldots n\ ;\ i \neq j}$.



(Figure 15)

For K_{n+1} o K_1, [In K_n o K_n the edges $v_{n+1}\, v_i$; $_{i = 1 \ldots n}$ and $v_{n+1}\, v_{n+1N}$ are added]

The minimum vertex covering sets with cardinality n + 1 are

$\{v_1, v_2 \ldots v_{i-1}, v_{i+1}, \ldots v_n, v_iN, v_{n+1}\}$; $_{i = 1 \ldots n;}$

$\{v_1, v_2 \ldots v_{n+1}\}$ and $\{v_1, v_2 \ldots v_n, v_{n+1}N\}$

Therefore, $c(G, n+1) = n+2$.

The vertex covering sets with cardinality $n+2$ for K_{n+1} o K_1 are the sets given in (1) by inserting v_{n+1} with each of the sets.

That is, $\{v_1, v_2 \ldots v_{i-1}, v_{i+1}, .. v_n, v_i^N, v_j^N, v_{n+1}\}$, $\{v_1, v_2 \ldots v_{i \ldots} v_n, v_i^N, v_{n+1}\}$; $i, j = 1 \ldots n$; $i \neq j$

Also, $\{v_1, v_2 \ldots\ldots, v_{n+1}, v_i^N\}$; $i = 1 \ldots n$ and $\{v_1, v_2 \ldots, v_n, v_{n+1}^N, v_j^N\}$; $j = 1 \ldots n+1$

Thus, the n^2 covering sets of K_n o K_1 [including the vertex v_{n+1}] and the additional $2n+1$ vertex covering sets from the family of covering sets of K_{n+1} o K_1 with cardinality $n+2$ form the covering sets of cardinality, $n+2$, for K_{n+1} o K_1.

Therefore, $c(G, n+2) = n(nC_1) + 2n + 1$.

Proceeding this way, $2n+1$ elements can be taken from $2(n+1)$ elements in $2(n+1)C_1$ ways, Therefore, $c(G, 2n+1) = 2(n+1)$, Also $c(G, 2n+2) = 1$.

Therefore, the vertex cover polynomial is

$$C(G, x) = (n+2)\, x^{n+1} + (n(nC_1) + 2n + 1)\, x^{n+2} + + 2(n+1)\, x^{2n+1} + x^{2n+2}$$

$$= (\overline{n+1} + 1)\, x^{n+1} + (n^2 + 2n + 1)\, x^{n+2} + \ldots + 2(n+1)\, x^{2n+1} + x^{2n+2}.$$

$$= (\overline{n+1} + 1)\, x^{n+1} + (n+1)(n+1)\, x^{n+2} + \ldots + 2(n+1)\, x^{2n+1} + x^{2n+2}$$

$$= (\overline{n+1} + 1)\, x^{n+1} + (n+1)[(n+1)C_1]\, x^{n+2} + . + 2((n+1)C_n).\, x^{2(n+1)-1} + (n+1)C_{n+1}\, x^{2(n+1)}$$

Hence the theorem. □

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