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The Concept of Measure of M-Divergence

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Abstract

M-divergence is a measure of the directed divergence of a probability distribution $P=(p_1, p_2,........., p_n)$ from another probability distribution $Q=(q_1, q_2,.........., q_n)$ when the probabilities in both distributions are monotonic increasing or monotonic decreasing.

Keywords: entropy, directed divergence, monotonocity.

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Introduction

In 1999, Kapur and Sharma [2] introduced M-entropy and later the same was discussed in Ph.D. Thesis of S. Sharma [4]. In 1951 Kullback-Leibler [3] introduced the measure of directed divergence

$$D(P:Q)=\sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i}$$

subject to

$$\sum_{i=1}^{n} p_i = 1$$

Kapur [1] defined measure of directed divergence

$$D_a(P:Q)=\sum_{i=1}^{n}(1+ap_i)\ln\frac{(1+ap_i)}{(1+aq_i)},\ \ a>0.$$

subject to

$$\sum_{i=1}^{n} p_i = 1$$

Here in the present paper we have introduced the concept of measures of M-Divergence corresponding to above measures of directed divergence.

The First Measure of M-Divergence

The first such measure is defined by

$$D_1(P:Q) = p_1 \ln\frac{p_1}{q_1} + (p_2 - p_1)\ln\frac{p_2 - p_1}{q_2 - q_1} + \ldots\ldots\ldots\ldots\ldots$$
$$+(p_n - p_{n-1})\ln\frac{p_n - p_{n-1}}{q_n - q_{n-1}} + (1 - p_n)\ln\frac{1 - p_n}{1 - q_n} \tag{1}$$

subject to

$$q_1 < q_2 < \ldots\ldots < q_n;\ p_1 < p_2 < \ldots\ldots < p_n. \tag{2}$$

Now

$$\frac{\partial D_1}{\partial p_1} = \ln\frac{p_1}{q_1} - \ln\frac{p_2 - p_1}{q_2 - q_1},\quad \frac{\partial^2 D_1}{\partial p_1^2} = \frac{1}{p_1} + \frac{1}{p_2 - p_1} > 0$$

$$\frac{\partial D_1}{\partial p_2} = \ln\frac{p_2 - p_1}{q_2 - q_1} - \ln\frac{p_3 - p_2}{q_3 - q_2},\quad \frac{\partial^2 D_1}{\partial p_2^2} = \frac{1}{p_2 - p_1} + \frac{1}{p_3 - p_2} > 0$$

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$$\frac{\partial D_1}{\partial p_n} = \ln\frac{p_n - p_{n-1}}{q_n - q_{n-1}} - \ln\frac{1 - p_n}{1 - q_n},\quad \frac{\partial^2 D_1}{\partial p_n^2} = \frac{1}{p_n - p_{n-1}} + \frac{1}{1 - p_n} > 0$$

$$\frac{\partial^2 D_1}{\partial p_i \partial p_{i+1}} = -\frac{1}{p_{i+1} - p_i}$$

$$\therefore \frac{\partial^2 D_1}{\partial p_i^2}.\frac{\partial^2 D_1}{\partial p_{i+1}^2} - \left(\frac{\partial^2 D_1}{\partial p_i \partial p_{i+1}}\right)^2 > 0$$

Thus $D_1(P:Q)$ is a convex function of $p_1, p_2, \ldots\ldots\ldots, p_n$.

Its minimum value subject to

$$\sum_{i=1}^{n} p_i = 1$$

is given by

$$\frac{p_2 - p_1}{p_1} = \frac{q_2 - q_1}{q_1}, \frac{p_3 - p_2}{p_2 - p_1} = \frac{q_3 - q_2}{q_2 - q_1}, \ldots\ldots\ldots\ldots,$$

$$\frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}} = \frac{q_n - q_{n-1}}{q_{n-1} - q_{n-2}}, \frac{p_n - p_{n-1}}{1 - p_n} = \frac{q_n - q_{n-1}}{1 - q_n}$$

which is satisfied if

$$p_1 = q_1, p_2 = q_2, \ldots\ldots\ldots\ldots, p_n = q_n.$$

i.e. $P = Q$ (3)

so that the minimum value of $D_1(P:Q)$ arises when $P = Q$ and $D_1(P;Q) \geq 0$.

We can use this $D_1(P:Q)$ as a measure of M-divergence when both p_i's and q_i's are monotonic increasing.

Thus when there are no constraints except the natural constraint $\sum_{i=1}^{n} p_i = 1$ and the inequality constraints $p_i \geq 0,\ 1 \geq p_i \geq p_{i-1}, i = 1,2,\ldots\ldots, n,$ the minimum M-divergence probability distribution is given by (3) and is same as the apriori distribution.

The Second Measure of M-Divergence

The second such measure is defined by

$$D_2(P:Q) = (1 + ap_i)\ln\frac{1 + ap_i}{1 + aq_i} + a(p_2 - p_1)\ln\frac{p_2 - p_1}{q_2 - q_1} + \ldots\ldots\ldots\ldots + \ldots\ldots \tag{4}$$

subject to

$$q_1 < q_2 < \ldots\ldots\ldots\ldots < q_n \ ; \ p_1 < p_2 < \ldots\ldots\ldots\ldots < p_n. \tag{5}$$

Now

$$\frac{\partial D_2}{\partial p_1} = a\ln\frac{1 + ap_1}{1 + aq_1} - a\ln\frac{p_2 - p_1}{q_2 - q_1}, \frac{\partial^2 D_2}{\partial p_1^2} = \frac{a^2}{1 + ap_1} + \frac{a}{p_2 - p_1} > 0$$

$$\frac{\partial D_2}{\partial p_2} = a\ln\frac{p_2 - p_1}{q_2 - q_1} - a\ln\frac{p_3 - p_2}{q_3 - q_2}, \frac{\partial^2 D_2}{\partial p_2^2} = \frac{a}{p_2 - p_1} + \frac{a}{p_3 - p_2} > 0$$

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$$\frac{\partial D_2}{\partial p_n} = a\ln\frac{p_n - p_{n-1}}{q_n - q_{n-1}} - a\ln\frac{p_{n+1} - p_n}{q_{n+1} - q_n}, \frac{\partial^2 D_2}{\partial p_n^2} = \frac{a}{p_n - p_{n-1}} + \frac{a}{p_{n+1} - p_n} > 0$$

$$\frac{\partial^2 D_2}{\partial p_i \partial p_{i+1}} = -\frac{a}{p_{i+1} - p_i}$$

$$\therefore \frac{\partial^2 D_2}{\partial p_i^2}\frac{\partial^2 D_2}{\partial p_{i+1}^2} - \left(\frac{\partial^2 D_2}{\partial p_i \partial p_{i+1}}\right)^2 > 0$$

Thus $D_2(P:Q)$ is a convex function of $p_1, p_2,........., p_n$.
Its minimum value subject to

$$\sum_{i=1}^{n} p_i = 1$$

is given by

$$\frac{p_2 - p_1}{1 + ap_1} = \frac{q_2 - q_1}{1 + aq_1}, \frac{p_3 - p_2}{p_2 - p_1} = \frac{q_3 - q_2}{q_2 - q_1}, ,.........., \frac{p_{n+1} - p_n}{p_n - p_{n-1}} = \frac{q_{n+1} - q_n}{q_n - q_{n-1}}$$

which is satisfied if

$$p_1 = q_1, p_2 = q_2,..........., p_n = q_n.$$

i.e. $P = Q$ (6)

so that the minimum value of $D_2(P:Q)$ arises when $P = Q$ and $D_2(P;Q) \geq 0$.

We can use this $D_2(P:Q)$ as a measure of M-divergence when both $p_i's$ and $q_i's$ are monotonic increasing.

Thus when there are no constraints except the natural constraint $\sum_{i=1}^{n} p_i = 1$ and the inequality constraints $p_i \geq 0, 1 \geq p_i \geq p_{i-1}, i = 1,2,......., n,$ the minimum M-divergence probability distribution is given by (6) and is same as the apriori distribution.

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