Modular Identities and Explicit Values of a New Continued Fraction of Ramanujan

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Abstract
In this paper, we study a new continued fraction of Ramanujan and find its modular identities and some explicit values.

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1. Introduction
Throughout the paper, we assume $|q| < 1$. As usual, for positive integers $n$ and any complex number $a$, we write

$$(a)_n := (a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j) \quad \text{and} \quad (a)_{\infty} := (a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n).$$ (1)

Ramanujan’s theta-functions are defined by

$$\phi(q) = 1 + 2 \sum_{k=1}^{\infty} q^{k^2} = \frac{(-q; q^2)_{\infty}(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}(-q^2; q^2)_{\infty}},$$ (2)

$$\psi(q) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}},$$ (3)

$$f(-q) = \sum_{k=0}^{\infty} (-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty} (-1)^k q^{k(3k+1)/2} = (q; q)_{\infty}.$$ (4)
Ramanujan recorded many $q$–continued fractions and some of their explicit values in his second notebook [3] and in his lost notebook [4]. The following beautiful continued fraction identity was recorded by Ramanujan in his second notebook and can be found in [1, p. 11, Entry 11]:

\[\frac{(-a)_\infty(b)_\infty - (a)_\infty(-b)_\infty}{(-a)_\infty(b)_\infty + (a)_\infty(-b)_\infty} = \frac{a - b}{1 - q} + \frac{(a - bq)(aq - b)}{1 - q^3} + \frac{q(a - bq^2)(aq^2 - b)}{1 - q^5} + \ldots\]

(5)

where either $q$, $a$, and $b$ are complex numbers with $|q| < 1$, or $q$, $a$, and $b$ are complex numbers with $a = bq^m$ for some integer $m$. Several elegant $q$-continued fractions can be expressed in terms of Ramanujan’s theta-functions. The most famous of them is the Rogers-Ramanujan continued fraction $R(q)$ defined by

\[R(q) := \frac{q^{1/5}}{1 - \frac{q}{1 + \frac{q^3}{1 + \frac{q^5}{1 + \ldots}}}}\]

(6)

The continued fraction $R(q)$ has a very beautiful and extensive theory almost all of which was developed by Ramanujan. An account of this can be found in in Chapter 32 of Berndt’s book [2].

Now, setting $a = q$ and $b = 0$ in (5), we obtain

\[\frac{(q; q)_\infty - (q; q)_\infty}{(q; q)_\infty + (q; q)_\infty} = \frac{q}{1 - q} + \frac{q^3}{1 - q^3} + \frac{q^5}{1 - q^5} + \ldots\]

(7)

In this paper, we study the Ramanujan’s continued fraction $L(q)$ defined by

\[L(q) := \frac{q}{1 - q} + \frac{q^3}{1 - q^3} + \frac{q^5}{1 - q^5} + \ldots\]

(8)

which is the right hand side of the equality (7).

In section 2, we prove some modular relations connecting $L(q)$ and $L(q^n)$. In section 3, we give some explicit evaluations of $L(q)$.

We end this introduction by recording some theta-function identities from [1, p. 40, Entry 25]:

\[\phi(q) + \phi(-q) = 2\phi(q^4),\]

(9)

\[\phi(q)\phi(-q) = \phi(-q^2),\]

(10)

\[\phi^2(q) + \phi^2(-q) = 2\phi^2(q^2),\]

(11)
2. Modular relations of $L(q)$

Theorem 2.1. We have

\[ L(q) := \frac{1 - \phi(-q)}{1 + \phi(-q)}. \]

Proof. Dividing numerator and denominator on left hand side of (7) by $(q; q)_{\infty}$, we obtain

\[ L(q) := \left( \frac{\frac{(-q; q)_{\infty}}{(q; q)_{\infty}} - 1}{\frac{(-q; q)_{\infty}}{(q; q)_{\infty}} + 1} \right) \quad (12) \]

From (1), we have

\[ ( - q ; q )_{\infty} = (q^2 ; q^2)_{\infty} / (q ; q)_{\infty}. \quad (13) \]

Employing (13) in (12) and using (4), we find that

\[ L(q) := \left( \frac{f(-q^2)}{f^2(-q)} - 1 \right) / \left( \frac{f(-q^2)}{f^2(-q)} + 1 \right) \quad (14) \]

Noting $\phi(-q) = ( f^2(-q) / f(-q^2) )$ from [1, p. 39, Entry 24], employing in (14) and simplifying, we complete the proof. ■

Corollary 2.2. We have

\[ \frac{1 - L(q)}{1 + L(q)} = \phi(-q). \]

Proof. Follows easily from Theorem 2.1. ■

From Corollary 2.2, it is obvious that if we have an identity connecting $\phi(-q)$ and $\phi(\pm q^n)$ then we can always find a modular relation between the continued fraction $L(q)$ and the continued fractions $L(\pm q^n)$. We give some examples in the theorem below.

Theorem 2.3. Let $u = L(q), v = L(-q), w = L(q^2), w = L(q^4), r = L(-q^2)$ and $s = L(-q^4)$, then

\begin{enumerate}
  \item $2s - u + su - v + sv - 2uv = 0,$
  \item $u + v - 2w - 2uvw + uw^2 + vw^2 = 0,$
  \item $2r - u + 2ru - r^2u + 2ru^2 - v + 2rv - r^2v - 4uv - 4r^2uv - u^2v$
  \hspace{1cm} + $2ru^2v - r^2u^2v + 2rv^2$
  \hspace{1cm} - $uv^2 + 2ruv^2 - r^2uv^2 + 2ru^2v^2 = 0,$
  \item $r - u - r^2u + u^2 + 3ru^2 + r^2u^2 + 2rv - 4uv - 4ruv - 4r^2uv$
  \hspace{1cm} + $2ru^2v + v^2 + 3rv^2 + r^2v^2$
  \hspace{1cm} - $uv^2 - r^2uv^2 + ru^2v^2 = 0.$
\end{enumerate}
Proof. To prove (i), (ii), and (iii), we employ Corollary 2.2 in (9), (10), and (11) respectively. To prove (iv), eliminating $\phi(-q)$ between (9) and (11), we arrive at

$$\phi^2(q) + 2\phi^2(q^4) - 2\phi(q)\phi(q^4) - \phi^2(q^2) = 0. \quad (15)$$

Employing Corollary 2.2 in (15), we complete the proof. ■

3. Explicit Evaluations of $L(q)$

Theorem 3.1. We have

(i) $L(e^{-n\pi}) = \frac{1 - \phi(-e^{-n\pi})}{1 + \phi(-e^{-n\pi})}$ and (ii) $L(-e^{-n\pi}) = \frac{1 - \phi(e^{-n\pi})}{1 + \phi(e^{-n\pi})}$.

Proof. We set $q := e^{-n\pi}$ and $q := -e^{-n\pi}$ in Theorem 2.1 to prove (i) and (ii), respectively. ■

In his first notebook, Ramanujan [3, Vol. I, p. 248] recorded many elementary values of $\phi(q)$. Particularly, he recorded $\phi(e^{-n\pi})$ and $\phi(-e^{-n\pi})$ for $n=1, 2, 4, 8, 1/2, \text{ and } 1/4$. Ramanujan also recorded non-elementary values of $\phi(e^{-n\pi})$ for $n= 3, 5, 7, 9, \text{ and } 45$. Proofs of these can be found in [2]. Yi [5] also evaluated $\phi(e^{-n\pi})$ for $n=1, 2, 3, 4, 5, \text{ and } 6$ and $\phi(-e^{-n\pi})$ for $n=1, 2, 4, 6, 8, 10, \text{ and } 12$. Noting from [2, p. 325, Entry 1(i) & (ii)], we have

$$\phi(e^{-\pi}) = a \text{ and } \phi(-e^{-\pi}) = 2^{-1/4}a, \quad (16)$$

where

$$a = \pi^{1/4}/\Gamma(3/4). \quad (17)$$

Employing (16) in Theorem 3.1, we obtain

$$L(-e^{-\pi}) = \frac{1 - a}{1 + a} \text{ and } L(e^{-\pi}) = \frac{2^{1/4} - a}{2^{1/4} + a}.$$